Math 374

Superstables

$c \in \text{Config}(G,q)$ is **superstable** if $c \geq 0$ and it has no nonempty legal set firings, i.e., $\forall\, \emptyset \neq S \subseteq \tilde{V}$, $\exists\, v \in \tilde{V}$ s.t.

$$c(v) < \text{outdeg}_S(v).$$

* Let $D \in \text{Div}(G)$ and write $D = c + kq$ with $c \in \text{Config}(G,q)$. Then D is q-reduced iff c is superstable.

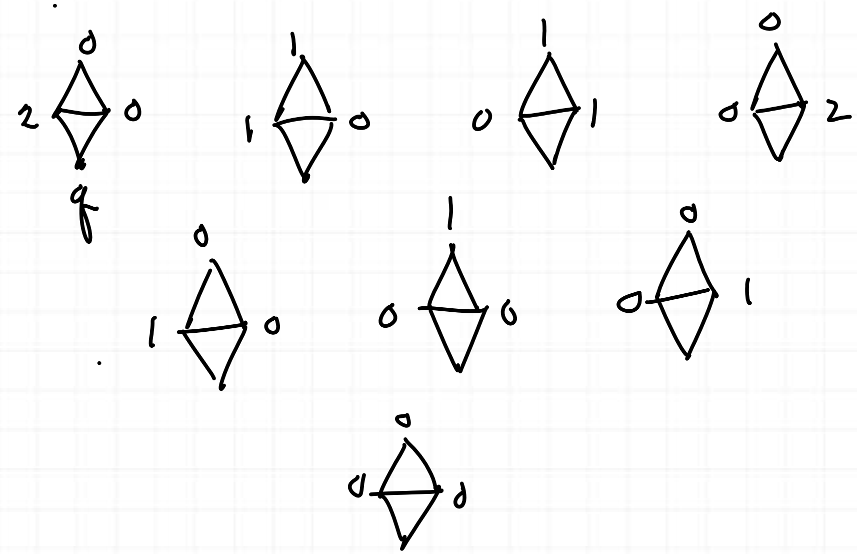
* By $\exists!$ of q-reduced divisors, each equivalence class in

$$\text{Config}(G)/\sim \;\cong\; \mathbb{Z}\tilde{V}/\text{im}\,\tilde{L}$$

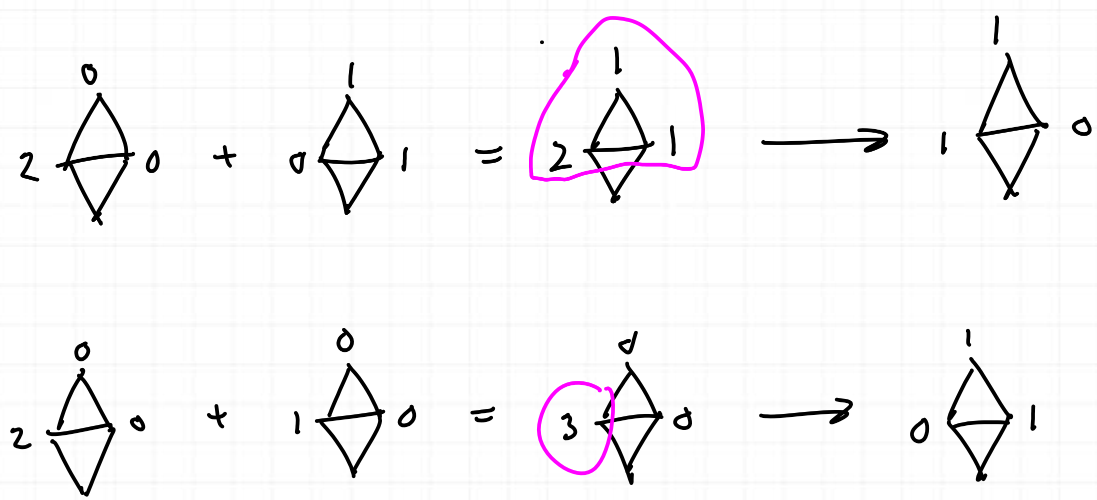
is represented by a unique superstable.

So the superstables form a group with the operation being addition followed by superstabilization.

Example



superstables



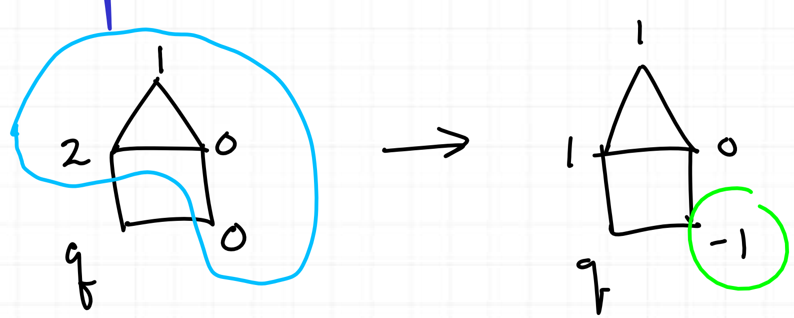
Examples of addition + stabilization

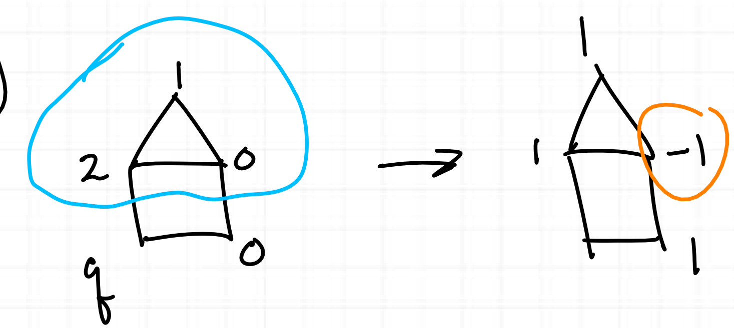
Dhar's Algorithm

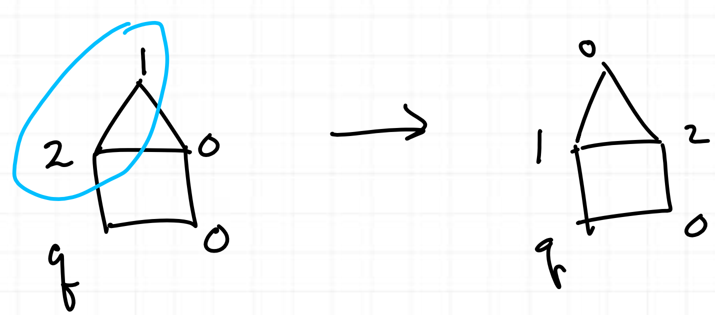
In order to superstabilize a configuration, it may seem that we would need to check the $2^{\#\tilde{V}}-1$ nonempty subsets to see if any are legal to fire. It turns out to be much faster.

* See Dhar's algorithm in the text.

Examples

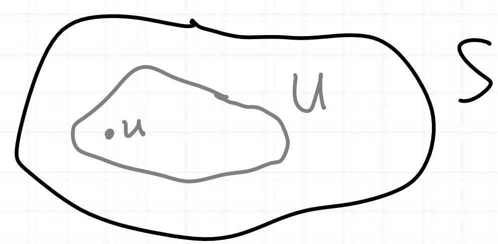
(1) 

(2) 

(3) 

Not superstable.

Pf/ The algorithm returns a set S that is legal to fire. If $S \neq \emptyset$, then clearly, the configuration is not superstable. Suppose $S = \emptyset$ and $\emptyset \neq U \subseteq \tilde{V}$. Since $S = \emptyset$, at some point in the algorithm, every element of U is removed from S. Let $u \in U$ be the first point removed and consider the step in the algorithm just before u is removed:

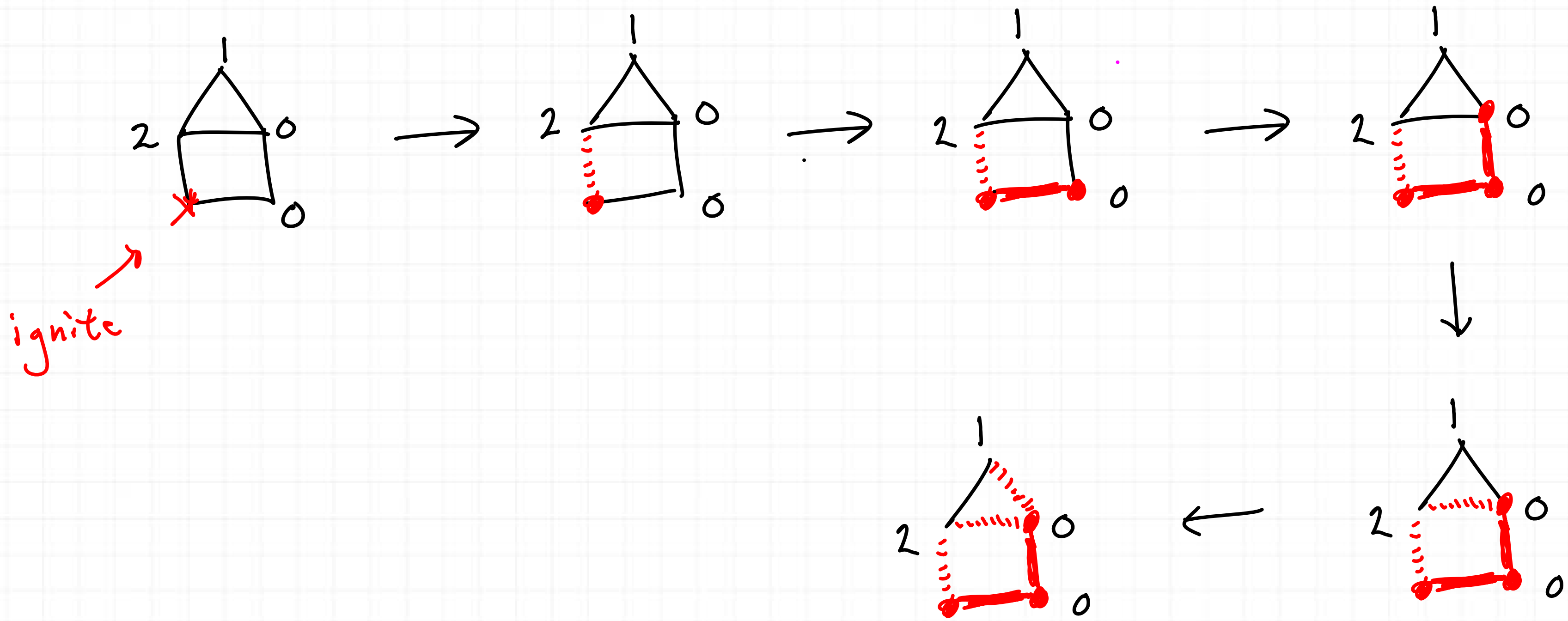


We have $\quad c(u) < \mathrm{outdeg}_S(u) \leq \mathrm{outdeg}_U(s)$.

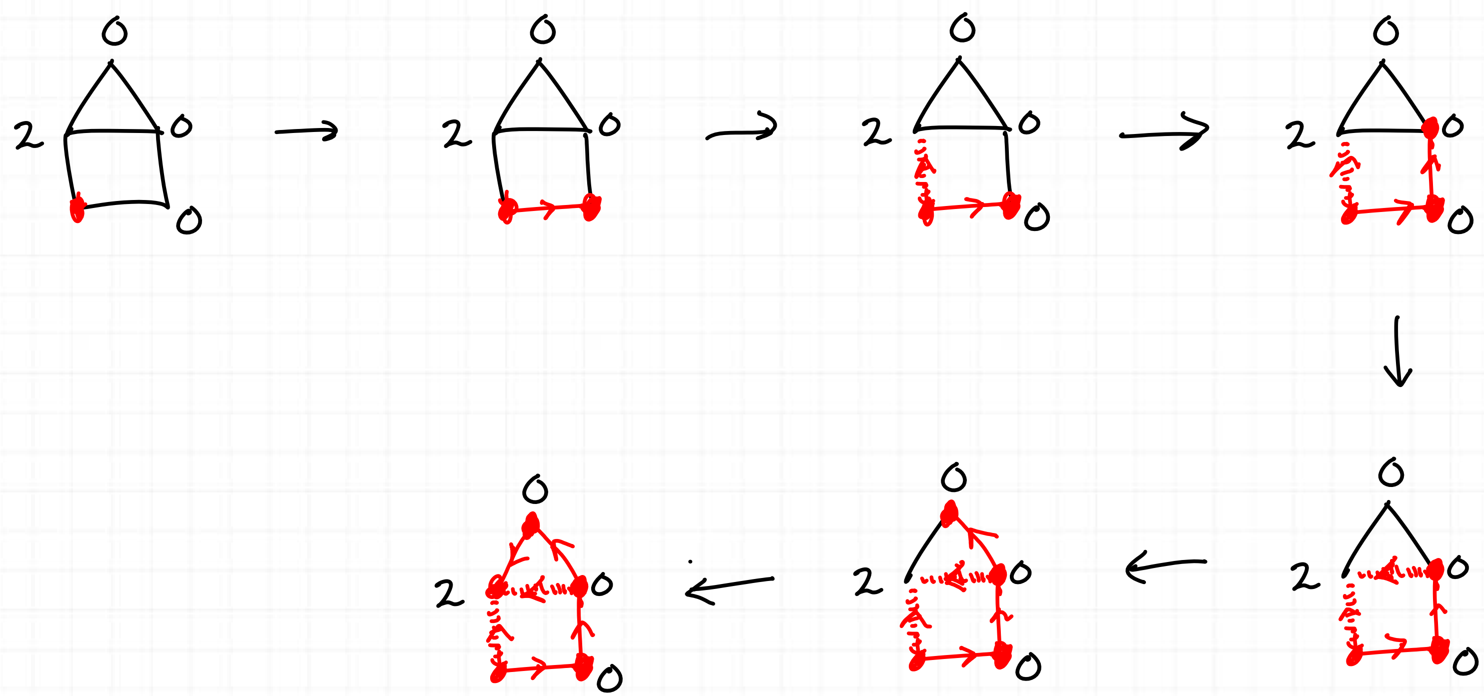
Thus, it is not legal to fire U. So c is superstable. $\square$

* See the book's interpretation of Dhar's algorithm in terms of fire fighters.

$c(v)$ = # firefighters on vertex v



The unburnt vertices gives a legal firing set.



All of the vertices are burnt. Therefore, the configuration is superstable.

* Note: For a superstable configuration, the burnt edges form a spanning tree.

* Open problem: How do systematize choices to get a bijection between superstables and spanning trees?

Once we have a system γ : superstables $\to$ spanning trees, what feature of $\gamma(c)$ corresponds with $\deg(c)$?