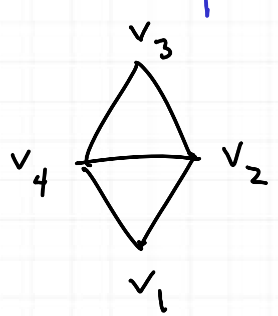
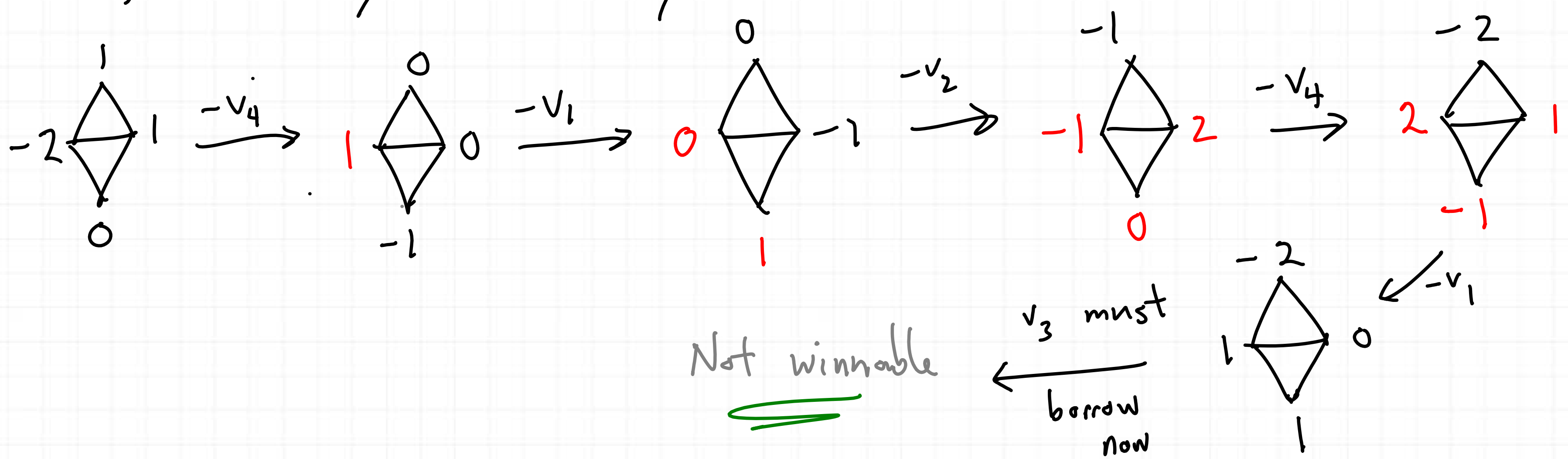
Math 374

Greedy algorithm

* To determine if D is winnable, just repeatedly choose a vertex in debt and borrow. There are two possibilities: ① at some point, each vertex has made a borrowing move, in which case, D is unwinnable, or ② you eventually win.

Example





Pf/ See text. Summary:

Suppose D is winnable. $\exists\, \sigma$ s.t. $D \xrightarrow{\sigma} E$ with $E \geq 0$.

By adding or subtracting $\vec{1}$, we may assume $\sigma \leq \vec{0}$ and $\exists v$ s.t. $\sigma(v) \neq 0$. Let $Z = \{v \in V : \sigma(v) = 0\}$. As long as D is in debt, pick an in-debt vertex u. Then necessarily $u \notin Z$. Modify D by borrowing at u, replace σ by $\sigma + u$, and add u to Z. This process must stop since $\sigma \leq 0$. At that point D has been transformed into an effective divisor (perhaps not E if the process stops before σ is transformed into $\vec{0}$.)

Now suppose D is unwinnable. Say $D = D_1 \xrightarrow{-v_1} D_2 \xrightarrow{-v_2} \cdots$ is an string of divisor obtained by borrowing at in-debt vertices $v_1, v_2, \ldots$ We have $D_i(v) \leq \max\{D(v), \deg_G(v)\} =: B_v$ $\forall v, i$. So $D_i(v) \leq \sum_v B_v =: B$. Since $\deg D_i = \deg D$ $\forall i$, we get

$$\deg D - nB \le D_i(v) \le nB$$

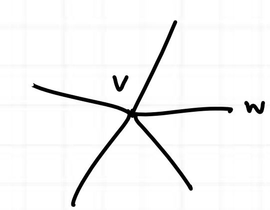
$\forall v, i$ where $n = \# B$. (If $D_i(u) < \deg D - nB$ for some u, then

$$\deg D = \deg D_i = \sum D_i(v) = D_i(u) + \sum_{v \neq u} D_i(v) < (\deg D - nB) + nB < \deg D.)$$

So $\{D_i\}$ is finite. Hence $\exists\, i < j$ s.t. $D_i = D_j$. Then letting $\sigma = -\sum_{k=i}^{j} v_k$, we get $D_i \xrightarrow{\sigma} D_j = D_i \Rightarrow \sigma = k\vec{1}$. Since $i \neq j$ and $\sigma \le 0$, we must have $k \neq 0$. Hence, going from D_i to D_j, every vertex has fired.

Prop. $\ker L = \mathbb{Z}\vec{1}$.

Pf/ Say $L\sigma = 0$. Pick v s.t. $\sigma(v) = \max\{\sigma(u) : u \in V\} =: k$. Then $L\sigma = 0 \Rightarrow (\deg_G v)\,\sigma(v) = \sum_{vw \in E} \sigma(w) \Rightarrow \sigma(w) = k$ $\forall w$. So $\sigma = k$. $\square$

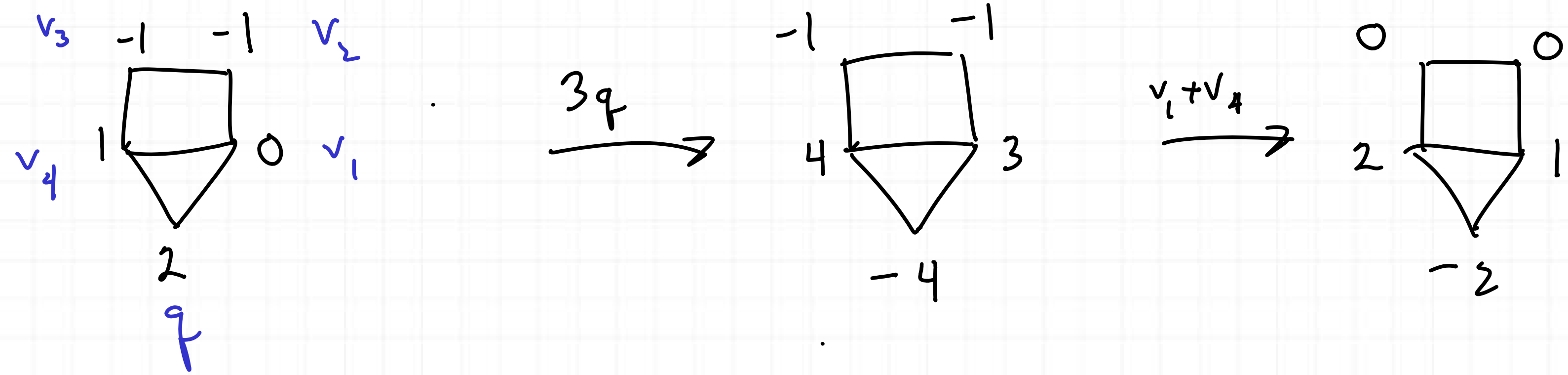


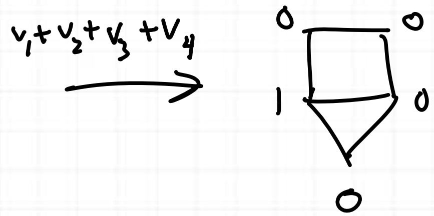
q-reduced divisors

Another algorithm for determining winnability:

* Fix $q \in V$. (The benevolent vertex.)
* Have q lend enough so that the other vertices may get out of debt by lending moves alone.
* After all vertices besides q are out of debt, the non-q vertices perform legal set lendings, i.e., set lendings (not including q) after which no non-q vertex is in debt.
* q is eventually out of debt, in which case the game is won. Otherwise the game is unwinnable.

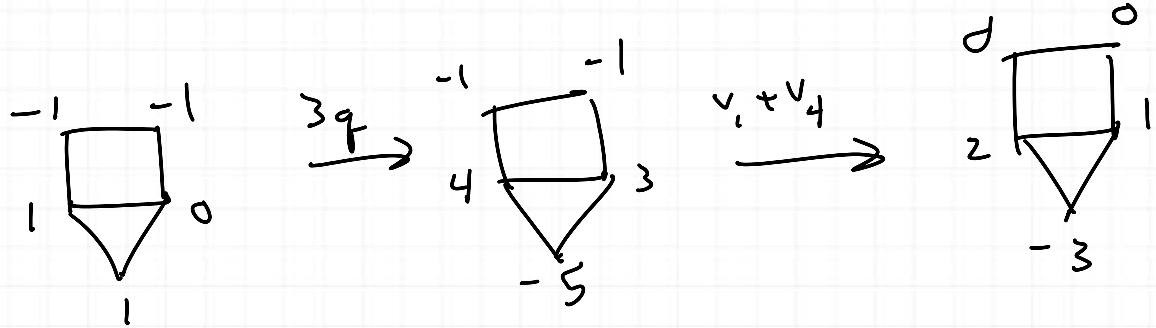
Example

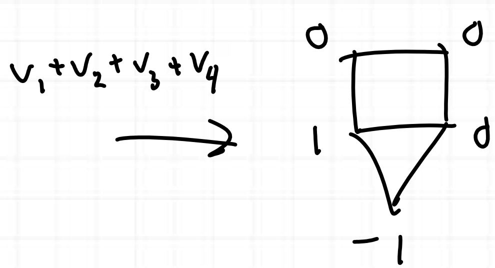




won!

On the other hand





Unwinnable.