Math 374

Determinantal divisors

Let M be an integer matrix with $\overbrace{\text{Smith normal form}}^{\text{SNF}}$ $\begin{bmatrix} s_1 & & & & & \\ & \ddots & & & & \\ & & s_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$.

So the invariant factors for M are $s_1, \ldots, s_k$.

The i^{th} determinantal divisor for M is $d_i := \gcd(i \times i \text{ minors of } M)$.

Since elementary integer row and column operations do not change the set of $i \times i$ minors (up to sign), it's easy to see from the SNF that $d_i = \prod_{j=1}^{i} d_j$.

Configurations

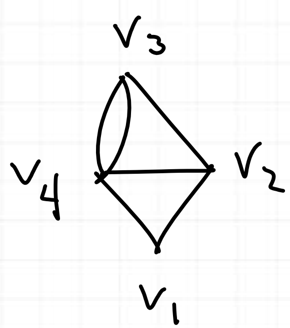
Fix $q \in V$ and set $\tilde{V} = V \setminus \{q\}$.

$\mathrm{Config}(G) := \mathbb{Z}\tilde{V} \subseteq \mathbb{Z}V = \mathrm{Div}(G)$

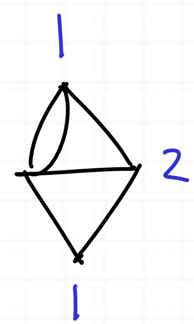
Define vertex firings for $v \in \widetilde{V}$ as before, just not recording the status of q.

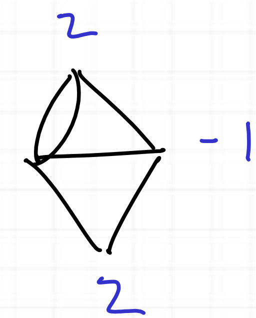
Example

Take $q = v_4$.



G

$c =$ 

$\xrightarrow{v_2}$ 

$(1, 2, 1)$ $\qquad$ $(2, -1, 2)$

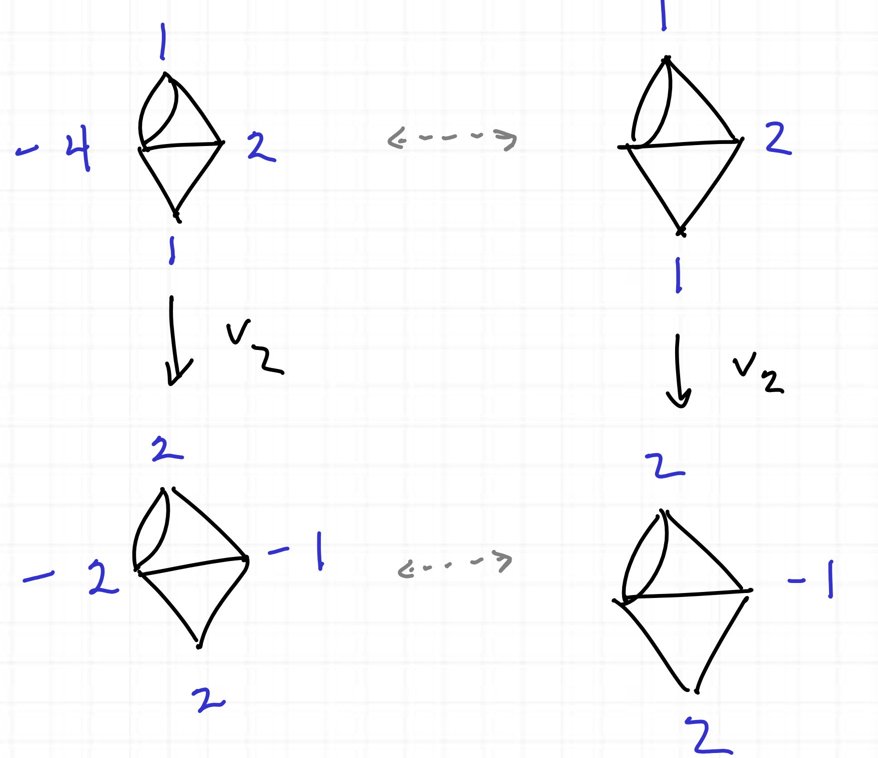
$$\mathrm{Jac}(G) = \mathrm{Div}^0(G)/\sim \;\xrightarrow{\;\cong\;}\; \mathrm{Config}(G)/\sim$$

$$D \longmapsto D|_{\tilde{v}}$$

$$c - \deg(c)\, q \longleftarrow c$$

Example

$\mathrm{Div}^0(G)$ $\quad\quad$ $\mathrm{Config}(G)$



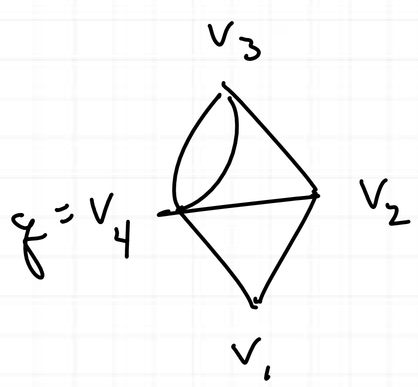
Reduced Laplacian (with respect to q)

$\tilde{L}$ is obtained from L by omitting the row and column corresponding to q. Let

$$\tilde{\mathcal{L}} := \operatorname{image}\left(\tilde{L} : \mathbb{Z}\tilde{V} \to \mathbb{Z}\tilde{V}\right).$$

Then

$$\operatorname{Config}(G)/\sim \;\approx\; \mathbb{Z}\tilde{V}/\tilde{\mathcal{L}}\,.$$



$$\tilde{L} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \end{array}$$

Note: L can be reconstructed from $\tilde{L}$ (for an undirected graph).

Since $\mathrm{Jac}(G) \cong \mathrm{Config}(G)/\sim \;\cong\; \mathbb{Z}\tilde{V}/\tilde{L}$, the invariant factors for $\mathrm{Jac}(G)$ can be computed from the SNF for $\tilde{L}$.

In the example from last time $\tilde{L} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 13 \end{bmatrix}$.)

Example

$$\tilde{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$_6$ = cycle graph on 6 vertices

The shaded 4×4 minor is $(-1)^4 = 1$.

it's easy to see $d_1 = d_2 = d_3 = d_4 = 1$. Direct calculations ws $d_5 = 6$. Hence, $s_1 = s_2 = s_3 = s_4 = 1$ and $s_5 = 6$. Therefore, $\mathrm{Pic}(C_6) = 6$.