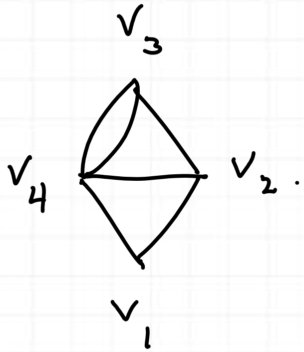
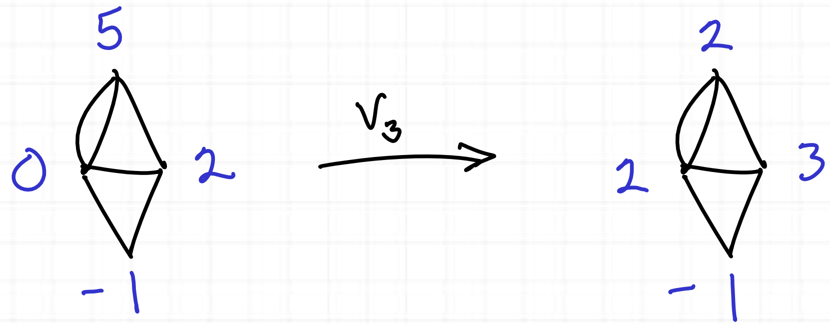
Math 374



G

$$L = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix} \end{array}$$



$$D = \begin{bmatrix} -1 \\ 2 \\ 5 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 \\ 2 \\ 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 2 \end{bmatrix} = D'$$

The Laplacian encodes firing rules.

$$\mathrm{Pic}(G) \cong \mathbb{Z}V / \mathcal{L} \quad \text{where } \mathcal{L} = \mathrm{image}(L : \mathbb{Z}V \to \mathbb{Z}V)$$

$$\mathrm{Pic}(G) \longrightarrow \mathbb{Z}^4 \Big/ \mathrm{im}\left(\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix}\right)$$

Which finite abelian group is this?

Idea: find 4×4 matrices, invertible over $\mathbb{Z}$, such that

$$P L Q = D = \begin{bmatrix} s_1 & & 0 \\ & s_2 & \\ 0 & & \ddots \end{bmatrix}$$

diagonal

$(\star)$

$$\begin{tikzcd}
\mathbb{Z}^4 \arrow[r, "L"] \arrow[d, "Q^{-1}"', "\cong"] \arrow[dr, phantom, "///"] & \mathbb{Z}^4 \arrow[r] \arrow[d, "\cong"', "P"] & \mathbb{Z}^4/L \arrow[r] \arrow[d, dashed, "\cong"'] & 0 \\
\mathbb{Z}^4 \arrow[r, "D"] & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^4/\mathrm{im}(D) \arrow[r] & 0
\end{tikzcd}$$

This isomorphism is induced by P.

Reason: (1) $\mathbb{Z}^4 / \text{im}(D)$ is clearly isomorphic to $\mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2} \times \cdots$

(2) $\text{Pic}(G) \cong \mathbb{Z}^4 / L \cong \mathbb{Z}^4 / \text{im}(D)$ by the above commutative diagram.

Smith normal form

★ Q: Column operations recorded in red

★ P: row operations recorded in blue

$$Q \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix} \xrightarrow{c_1 + c_2 \to c_1} \begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 3 & -1 & -1 \\ -1 & -1 & 3 & -2 \\ -2 & -1 & -2 & 4 \end{bmatrix} \xrightarrow[c_1 + c_4 \to c_4]{c_1 + c_2 \to c_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & -1 & 1 \\ -1 & -2 & 3 & -3 \\ -2 & -3 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow[\substack{r_1 + r_3 \to r_3 \\ 2r_1 + r_4 \to r_4}]{-2r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -1 & 1 \\ 0 & -2 & 3 & -3 \\ 0 & -3 & -2 & 2 \end{bmatrix} \xrightarrow{c_4 \leftrightarrow c_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 5 \\ 0 & -3 & 3 & -2 \\ 0 & 2 & -2 & -3 \end{bmatrix} \xrightarrow[-5c_2 + c_4 \to c_4]{c_2 + c_3 \to c_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 13 \\ 0 & 2 & 0 & -13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -4 \\ 1 & 1 & 1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -5 \end{bmatrix}$$

$$\xrightarrow[-2r_2+r_4\to r_4]{3r_2+r_3\to r_3}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&13\\0&0&0&-13\end{bmatrix}\xrightarrow[c_3\leftrightarrow c_4]{r_3+r_4\to r_4}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&13&0\\0&0&0&0\end{bmatrix}=D$$

$$\to\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-5&3&1&0\\6&-2&0&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-5&3&1&0\\1&1&1&1\end{bmatrix}=P$$

$$\longrightarrow\begin{bmatrix}1&1&-4&1\\1&1&-3&1\\0&0&0&1\\0&1&-5&1\end{bmatrix}=Q$$

$$\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-5&3&1&0\\1&1&1&1\end{bmatrix}\begin{bmatrix}2&-1&0&-1\\-1&3&-1&-1\\0&-1&3&-2\\-1&-1&-2&4\end{bmatrix}\begin{bmatrix}1&1&-4&1\\1&1&-3&1\\0&0&0&1\\0&1&-5&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-5&3&1&0\\1&1&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\2&1&0&0\\-1&-3&13&0\\-2&2&-13&0\end{bmatrix}=D. \quad \checkmark$$

Therefore, $$\mathrm{Pic}(G)\cong\frac{\mathbb{Z}^4}{\mathrm{im}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&13&0\\0&0&0&0\end{pmatrix}}\cong\mathbb{Z}/1\cdot\mathbb{Z}\times\mathbb{Z}/1\cdot\mathbb{Z}\times\mathbb{Z}/13\cdot\mathbb{Z}\times\mathbb{Z}/0\cdot\mathbb{Z}\cong\mathbb{Z}_{13}\times\mathbb{Z}.$$

As can be seen from the commutative diagram $\star$, the isomorphism is given by the matrix P:

$$
\begin{array}{ccccc}
\mathrm{Pic}(G) & \longrightarrow & \mathbb{Z}_1 \times \mathbb{Z}_1 \times \mathbb{Z}_{13} \times \mathbb{Z} & \longrightarrow & \mathbb{Z}_{13} \times \mathbb{Z} \\
(w,x,y,z) & \longmapsto & \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -5 & 3 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = (w, -2w+x, -5w+3x+y, w+x+y+z) & \longmapsto & (-5w+3x+y, w+x+y+z)
\end{array}
$$