Math 374

☆ HW due Friday

The dollar game

* solution to exercise 1.4

Vocabulary

graph: $G = (V, E)$. multiset of edges

divisors: $\mathrm{Div}(G) = \mathbb{Z}V = \{\sum_{v \in V} a_v v : a_v \in \mathbb{Z}\}$.

degree: If $D = \sum a_v v$, then $\deg(D) = \sum a_v$.

lending move by $v \in V$: $D \xrightarrow{v} D - \sum_{vw \in E} (v - w)$.

borrowing move by v: $D \xrightarrow{-v} D + \sum_{vw \in E} (v - w)$.

Abelian property
(diamond lemma)

$$\begin{tikzcd} & D \arrow[dl, "v"'] \arrow[dr, "w"] & \\ D' \arrow[dr, "w"'] & & D'' \arrow[dl, "v"] \\ & D''' & \end{tikzcd}$$

set firings: $W \subseteq V$ $\quad D \xrightarrow{W} D'$ if D' results from firing all vertices in W.

Prop. Lending and borrowing moves do not change the degree of a divisor.

Picard group

$D, D' \in \mathrm{Div}(G)$

* D is linearly equivalent to D', denoted $D \sim D'$, if D' may be obtained from D by a sequence of lending and borrowing moves.*

* $\sim$ is an equivalence relation

* if D' is obtainable through a sequence of lending and borrowing moves, then it's obtainable through a sequence of lending moves alone.

* divisor class $\quad [D] = \{ D' \in \mathrm{Div}(G) : D' \sim D \}$

* Picard group $\quad \mathrm{Pic}(G) = \mathrm{Div}(G)/\sim$.

* $\mathrm{Div}^k(G) := \{ D \in \mathrm{Div}(G) : \deg(D) = k \}$

Converse does not hold. Example?

* $\mathrm{Div}^0(G)$ is a subgroup of $\mathrm{Div}(G)$

* $\mathrm{Jac}(G) := \mathrm{Div}^0(G)/\sim$. This makes sense since $D \sim D' \Rightarrow \deg(D) = \deg(D')$

* Complete linear system

$$|D| := \{ D' \in \mathrm{Div}(G) : D' \sim D \text{ and } D' \geq 0 \}.$$

* D is winnable if $|D| \neq \emptyset$.

* $\mathrm{Pic}(G) \xrightarrow{\sim} \mathbb{Z} \times \mathrm{Jac}(D) \qquad (\text{Fix } q \in V)$

$\quad [D] \longmapsto (\deg(D), [D - (\deg D) q])$

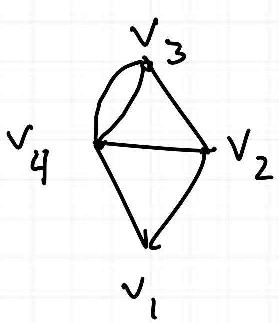
* Laplacian matrix   Order vertices: $v_1, \dots, v_n$

$$D = \mathrm{diag}(\deg_G(v_1), \dots, \deg_G(v_n)).$$

A = adjacency matrix ; A_{ij} = # edges $v_j v_i \in E$

$$L = D - A.$$

Example



$$L = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -2 \\ 0 & -1 & -2 & 4 \end{bmatrix} \end{array}$$

☆ The columns of L encode vertex borrowing moves.