Math 374 HW 10, due Monday, April 20

1. Let $G = (V, E, s)$ be an Eulerian sandpile graph. Prove that

$$\sum_{c \in \mathcal{S}(G)} \beta_v(c) = \#\mathcal{S}(G)$$

for each $v \in V$. Hence, the average burst size at v over all recurrents is 1.

2. Let G be the house graph, and consider the closed Markov chain on G.

   (a) Compute the Tutte polynomial of G.

   (b) Compute $\lim_{\deg(D_0) \to -\infty} \mathbb{P}_{D_0}(\deg D_\tau = n)$.

   (c) Use Sage to simulate the closed Markov chain on G starting at $D_0 = \vec{0}$.

      i. Run the simulation 1000 times and report the degrees of the threshold divisors.

      ii. Do your results agree with your answer to part (b) of this problem?

      iii. Try your simulation again starting with a D_0 with negative degree. What happens?

3. Table 1 lists the sorted stable states and sorted minimal threshold states for the closed Markov chain on K_4. For a state D, let $\lambda(D)$ be the number of paths in the chain from 0000 to D. For instance $\lambda(0001) = 1$.

degree	divisor			threshold state	
0	0000				
1	0001				
2	0011	0002			
3	0111	0012			
4	1111	0112	0022		
5	1112	0122			
6	1122	0222		0123	
7	1222			1123	0223
8	2222			1223	
9				2223	

Table 1: Sorted stable and minimal threshold states for K_4 by degree.

For each D in the table, if $\deg(D) = d \geq 1$, then $\lambda(D)$ may be expressed as an integer linear combination of $\lambda(D')$s for *stable* states D' in the table with degree $d - 1$. For instance, $\lambda(0011) = 2 \cdot \lambda(0001)$ and $\lambda(0002) = 1 \cdot \lambda(0001)$. We also have

$$\lambda(0111) = 3 \cdot \lambda(0011) + 1 \cdot \lambda(0002)$$

since 0111 may be obtained from 0011, 0101, 0110, and 2000 by adding a grain of sand and stabilizing. In class, we showed $\lambda(0112) = \lambda(0111) + 3 \cdot \lambda(0012)$. Ultimately, we would like to compute λ for each of the threshold states, from which we may compute the threshold density $\zeta_{\vec{0}}(K_4)$.

Crowd sourcing: please compute $\lambda(D)$ as a linear combination of *stable* $\lambda(D')$s with $\deg(D') = \deg(D) - 1$ for the states D appearing next to your initials, below:

MB: 1111, 0112, 2222

RT: 0022, 1112, 0223

NT: 0122, 1122, 1123

AS: 0222, 0123, 2223

AL: 1112, 1123, 0122

EM: 0223, 2222, 1122

DM: 1223, 2223, 0123

DT: 0112, 0022, 0222