Math 374 HW 9, due Monday, April 13

1. **Tutte polynomial.** Let $G = (V, E)$ be an undirected multigraph. We defined the Tutte polynomial to be

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{\text{rank}(E)-\text{rank}(A)} (y-1)^{\#A-\text{rank}(A)} \tag{1}$$

where $\text{rank}(A)$ is the size of a maximal forest of the graph (V, A). Thus, $\text{rank}(A) = \#V - k(A)$, where $k(A)$ is the number of components of (V, A).

Define $t_G(y) := T(G; 1, y)$. Recall that in class we showed that if G is connected, then its stationary density is

$$\zeta(G) := \frac{1}{\#V}\left(\#E + \frac{t'_G(1)}{t_G(1)}\right).$$

(a) Use equation (1) to verify that in the case G is connected, $T(G; 1, 1)$ is the number of spanning trees of G.

(b) Let G be K_4 with one edge deleted. Using equation (1), verify that $T(G; 1, y) = 4+3y+y^2$. Identify the sets A that contribute to nonzero terms.

(c) For arbitrary connected G show that $t'_G(1)$ is the number of *spanning unicycles* of G, where a *unicycle* is a subgraph having a single cycle. (So a spanning unicycle is a subgraph obtained from a spanning tree by adding an edge, or equivalently, is a subgraph with $\#V$ vertices and $\#V$ edges.)

(d) Let G be the graph with two vertices and $k+1$ edges joining those vertices.

   i. Use deletion and contraction to compute the Tutte polynomial of G.

   ii. Use this Tutte polynomial to compute the stationary density of G.

(e) Use deletion and contraction to compute the Tutte polynomial of the cycle graph, C_n.

2. **Trees.** Let T be a tree on n vertices. Let $\zeta(T)$ be its stationary density, and let $\zeta_\tau(\vec{0})$ be the threshold density starting from the all-zero divisor. Show that

$$\zeta(T) = \zeta_\tau(\vec{0}) = 1 - \frac{1}{n}.$$

3. **Cycle graphs.** Let C_n be the cycle graph on n vertices. Let $\zeta(C_n)$ be its stationary density, and let $\zeta_\tau(\vec{0})$ be the threshold density starting from the all-zero divisor. Show that

$$\zeta(C_n) = \zeta_\tau(\vec{0}) = 1 + \frac{1}{n^2}.$$