Problems for Chapter 14

14.1. Characterize all graphs with Jacobian group isomorphic to $\mathbb{Z}_4$. (Do not count graphs obtained from others by attaching a tree at a vertex. In other words, only graphs with no vertices of degree 1.)

14.2. Let M be an $n \times n$ matrix, and suppose the rows of M sum to the zero vector. Let $M^{(ij)}$ be the matrix obtained by removing the i-th row and j-th column from M. Then

$$(-1)^{i+j} \det M^{(ij)} = \det M^{(jj)}.$$

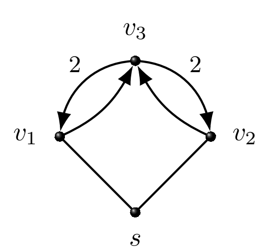
14.3.

(a) Show that

$$\det \begin{pmatrix} x & y & y & \dots & y \\ y & x & y & \dots & y \\ \vdots & & \ddots & & \vdots \\ y & y & y & \dots & x \end{pmatrix} = (x-y)^{n-1}(x+(n-1)y).$$

(b) Use the matrix-tree theorem to prove Cayley's theorem: the number of trees on n labeled vertices is n^{n-2}. (Note that "tree" in this case means a spanning tree of the complete graph on n vertices. The labels are mentioned to distinguish between isomorphic trees, i.e., trees isomorphic as graphs.)

14.4. Find all directed spanning trees into s in the following graph, checking for agreement with the matrix-tree theorem. Note that two of the edges have weight 2.



14.5. Let G be an undirected multigraph with n vertices, and let L be its Laplacian matrix. Let J be the $n \times n$ matrix whose entries are all 1s. Prove Proposition 14.6 using the hints below.

Let M be any $n \times n$ matrix, and let $M^{(ij)}$ be the matrix obtained from M by removing its i-th row and j-th column. The *adjugate* of M in the $n \times n$ matrix, $\operatorname{adj}(M)$, defined by $\operatorname{adj}(M)_{ij} := (-1)^{i+j} \det M^{(ji)}$; so the adjugate is the transpose of the cofactor matrix. Some well-known properties of the adjugate are: (i) $M \operatorname{adj}(M) = \det M$, and (ii) if N is another $n \times n$ matrix, $\operatorname{adj}(MN) = \operatorname{adj}(N) \operatorname{adj}(M)$.

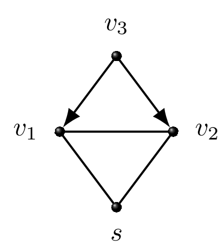
(a) Prove that
$$(n\, I_n - J)(L + J) = n\, L.$$
(b) Prove that $\operatorname{adj}(n\, I - J) = n^{n-2}$.

(c) Prove that $n^{-2} \det(L + J)$ is the number of spanning trees of G.

(d) * Does $n^{-2} \det(L + J)$ count directed spanning trees in the case of a directed multigraph?

14.6. Figure 6 starts with the configuration $(2, 1, 0)$ and an initial tree. By redrawing Figure 6 starting, instead, with the identity configuration, show that the final tree is the same as the initial tree.

14.7. Consider the following graph G with sink s:



(a) Find all recurrent configurations on G, indicating the identity configuration.

(b) Use the matrix-tree theorem to determine the number of directed spanning trees directed into the sink.

(c) **Rotor routers.** Choose a generator, c, for the sandpile group of G. The rotor-router action, $T \mapsto c \cdot T$, permutes the spanning trees of G. Describe this permutation by drawing a directed graph with the spanning trees of G as vertices and edges $(T, c{\cdot}T)$.

Better: move this to SNF chapter: compute the sandpile group.

14.8. Let $K_{m,n}$ be the *complete bipartite graph*. Its vertex sets is the disjoint union of two sets, U and V, of sizes m and n, respectively. The edge set consists of all pairs $\{u, v\}$ such that $u \in U$ and $v \in V$. Show that the number of spanning trees of $K_{m,n}$ is $n^{m-1}m^{n-1}$.