Problems for Chapter 5

5.1. Before beginning the Cori-Le Borgne algorithm (Algorithm 6), we fix a vertex q, and then at step 7 we always reduce with respect to that vertex. What would happen if each time we reached step 7 we were allowed to perform the reduction with respect to an arbitrarily chosen vertex? Would the algorithm still be valid?

5.2. Draw a diagram as in Figure 7 in order to determine the rank of the divisor $D = (0, 0, 0, 1, 1, 4, 6, 6, 9, 22)$ on K_{10}.

5.3. Find the ranks of all divisors on K_4. Let $p_1, \ldots, p_5$ be the increasing parking functions of length 3, and define the superstables $c_i := p_i - \vec{1}$ for each i. Define $D_i := c_i - \deg(c_i)q$ for $i = 1, \ldots, 5$. For each $d \in \mathbb{Z}$, the divisor classes of the $D_i + dq$ and of all the divisors obtained from them by permuting the components of the c_i are exactly the elements of $\mathrm{Pic}^d(D)$. In other words, up to symmetry and linear equivalence, the divisors of degree d are exactly $D_i + dq$ for $i = 1, \ldots 5$. Make a table showing the rank of $D_i + dq$ as i and d vary. Use the Cori-Le Borgne diagrams of Section 5.2.2 to compute the ranks.

Problems for Chapter 6

6.1. If D is a divisor on a tree, show that $r(D) = \deg(D)$ in two ways: (i) directly from the definition of rank, and (ii) from Riemann-Roch. (Note that a graph is a tree if and only if its genus is 0.

6.2. Let v be a vertex on a graph G of genus g. Show that

$$r(v) = \begin{cases} 1 & \text{if } g = 0\text{, i.e., } G \text{ is a tree,} \\ 0 & \text{if } g > 0. \end{cases}$$

6.3. Use the Riemann-Roch theorem to determine the rank of an arbitrary divisor on the cycle graph with n vertices, C_n.

6.4. Let $\mathcal{N}$ denote the maximal unwinnable divisors on a graph G, and let K be the canonical divisor of G. By Corollary 6.3, there is an involution $\iota \colon \mathcal{N} \to \mathcal{N}$ given by $\iota(N) = K - N$.

(a) Illustrate this involution for the graph G of Figure 1. List all maximal unwinnables in q-reduced form.

(b) Describe this involution for the complete graph, K_n.

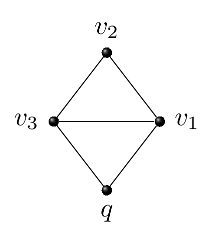

Figure 1. Graph G for Problem 6.4.