Problems for Chapter 3

3.1. Is the an algorithm for the dollar game based on benevolence as follows? As long as some vertex is in debt, a vertex that is not in debt lends. Repeat until everyone is out of debt or until a state is reached at which continuing would force each vertex to have made a lending move, at which point the game is certifiably unwinnable. Prove or disprove.

3.2. Let C_4 be the cycle graph with vertices v_1, v_2, v_3, v_4 listed in order around the cycle. Let $D = -2v_1 - 2v_2 + 2v_3 + 3v_4 \in \mathrm{Div}(C_4)$.

(a) Find the firing script, σ, determined by the greedy algorithm for the dollar game.

(b) Let L be the Laplacian for C_4. Verify that $D - L\sigma$ is effective.

3.3. Let C_n be the cycle graph with vertices $v_1, \ldots, v_n$, listed in order around the cycle. Suppose $n = 2m$ with $m > 1$, and let $D = -2v_1 + 2v_{m+1} \in \mathrm{Div}(C_n)$. What is the firing script produced by the greedy algorithm for the dollar game applied to D?

3.4. Let G be the house graph picture in Figure 3 in the Problems for Chapter 2. Let $q = v_1$, and compute linearly equivalent q-reduced divisors for the following divisors on G:

(a) $D_1 = (-3, 2, 4, -2, 1)$

(b) $D_2 = (2, 1, -5, 2, 2)$

(c) $D_3 = (0, -2, -2, 0, 0)$.

3.5. Consider a variation of the protocol for parking cars described in §3.4.1. There are still n cars, $C_1, \ldots, C_n$, but this time there is one extra parking space, numbered $n+1$, and the spaces are arranged in a circle. Car C_i prefers to park in space $p_i \in \{1, \ldots, n+1\}$. Other than that, the rules are essentially the same: each car in turn drives to its preferred spot and parks there if possible. Otherwise, it drives on to the next available spot. Since the spaces are arranged in a circle, each car will eventually park. Call these preference lists *circular parking functions*.

Credit Matthias Beck with this terminology. Is it his?

(a) After the cars park according to a given circular parking function, there is one empty parking space. Show that the number of circular parking functions that leave space i empty is the same as the number that leave space 1 empty, for each i.

(b) Show that a circular parking function is an actual parking functions if and only if it leaves space $n+1$ empty.

(c) Conclude that the number of ordinary parking functions of length n is $(n+1)^{n-1}$.

3.6. Let $D \in \mathrm{Div}(G)$, and fix a source vertex $q \in V$. The proof of the existence and uniqueness theorem for q-reduced divisors guarantees the existence of a firing script σ such that $D \xrightarrow{\sigma} D'$ where $D'(v) \geq 0$ for all $v \neq q$. Using this fact, mimic the proof of the validity of Algorithm 1, Chapter 3, to verify that the greedy algorithm in Algorithm 4 brings the non-source vertices of D out of debt.

Algorithm 4 Greedy algorithm for problem 3.6.

1: INPUT: $D \in \mathrm{Div}(G)$.
2: OUTPUT: $D' \sim D$ such that $D'(v) \geq 0$ for all $v \in \widetilde{V}$.
3: **initialization:** $D' = D$.
4: **while** $D'|_{\widetilde{V}} \ngeq 0$ **do**
5: choose $v \in \widetilde{V}$ such that $D'(v) < 0$
6: modify D' by performing a borrowing move at v
7: **return** D'

3.7. Let $c \in \mathrm{Config}(G, q)$. Fixing an ordering of the vertices, let $\widetilde{L}$ be the reduced Laplacian of G and identify c with an integer vector in $\mathbb{Z}^{n-1}$, as usual. Let

$$\sigma := \lfloor \widetilde{L}^{-1}\, c \rfloor \in \mathbb{Z}^{n-1}$$

be the integer vector obtained from $\widetilde{L}^{-1}\, c$ by taking the floor of each of its components. Define $c' = c - \widetilde{L}\, \sigma$. Prove that $|c'(v)| < \deg_G(v)$ for all $v \in \widetilde{V}$.

We should indicate our sources for some of these problems.

Should we have a **Notes** section at the end of some chapters?

its incident edges. In any case, add v' to S. It is still the case that v is the unique source vertex in S. Repeat until $S = V$ to obtain an acyclic orientation with unique source v. $\square$

Definition 4.12. For any orientation $\mathcal{O}$ on a graph G, denote by $\mathcal{O}^{\text{rev}}$ the *reversed orientation* obtained by reversing the direction of each edge in $\mathcal{O}$. Define the *canonical divisor* of G to be the divisor

$$K := \mathbf{D}(\mathcal{O}) + \mathbf{D}(\mathcal{O}^{\text{rev}}).$$

Note that for every $v \in V(G)$

$$K(v) = (\text{indeg}_{\mathcal{O}}(v) - 1) + (\text{outdeg}_{\mathcal{O}}(v) - 1) = \deg_G(v) - 2,$$

so that the canonical divisor depends only on the graph G and not on the orientation $\mathcal{O}$.

Exercise 4.13. Show that $\deg(K) = 2g - 2$, where $g = |E| - |V| + 1$ is the genus of the graph G.

The canonical divisor of a graph will play a key role in the Riemann-Roch theorem of chapter 6.

Problems for Chapter 4

4.1. If D is an unwinnable divisor, why must there exist a maximal unwinnable divisor D' such that $D \leq D'$.

4.2. Let G be the house graph from Problem 2.5, and take $q = v_1$. Find all maximal superstables on G and their corresponding acyclic orientations.

where $\widetilde{1} \in \mathbb{Z}^{n-1}$ denotes the all ones vector. Thus, in the new coordinates, our system of inequalities becomes

$$\begin{array}{rcl} \widetilde{y} & \geq & 0 \\ \widetilde{1} \cdot \widetilde{y} & \leq & d, \end{array}$$

where $d = a_n + \widetilde{1} \cdot \widetilde{a} = \deg(D)$ is the degree of the original divisor, D. This is clearly a bounded simplex in $\mathbb{R}^{n-1}$. But the original coordinates $\widetilde{x}$ are obtained from $\widetilde{y}$ by an affine-linear transformation, so $\widetilde{P}$ is also bounded. $\square$

Here is a nice paper by De Loera, et. al. explaining an implementation of the Barvinok-Pommersheim algorithm: www.sciencedirect.com/science/article/pii/S0747717104000422

5.2. The rank function.

You might think that "degree of winnability" should be measured by the size of complete linear systems, so that D is "more winnable" than D' if $\#|D| > \#|D'|$. But it turns out that a less obvious notion is more fruitful: instead of demanding an exact determination of $|D|$, we will introduce a quantity $r(D) \in \mathbb{Z}$ that measures "robustness of winnability." To begin, define $r(D) = -1$ if D is an unwinnable divisor. That is:

$$r(D) = -1 \iff |D| = \emptyset.$$

Next, define $r(D) = 0$ if D is *barely winnable* in the sense that D is winnable, but there exists a vertex $v \in V$ such that $D - v$ is unwinnable. That is, $r(D) = 0$ if and only if the winnability of D can be destroyed by a single vertex losing a dollar. In general, for $k \geq 0$ we define

$$r(D) \geq k \iff |D - E| \neq \emptyset \qquad \text{for all effective } E \text{ of degree } k.$$

In words: $r(D) \geq k$ if and only if the dollar game on G is winnable starting from all divisors obtained from D by removing k dollars from the graph. It follows that $r(D) = k$ if and only if $r(D) \geq k$ and there exists an effective divisor E of degree $k+1$ such that $D - E$ is not winnable.

Exercise 5.2. Show that $r(D) \leq \max\{-1, \deg(D)\}$ for all divisors D.

Exercise 5.3. Show that if D is a divisor of degree 0, then $r(D) = 0$ if and only if D is principal.

Exercise 5.4. Show that $r(D) \leq r(D+v) \leq r(D)+1$ for all $D \in \mathrm{Div}(G)$ and all $v \in V$. That is, adding one dollar to a divisor can increase its rank by at most one.

Exercise 5.5. Show that if $r(D), r(D') \geq 0$, then $r(D+D') \geq r(D) + r(D')$.

Proposition 5.6. *Suppose that G is a d-edge connected graph and $E \in \mathrm{Div}_+(G)$ is an effective divisor of degree less than d. Then $r(E) = \min\{E(v) : v \in V(G)\}$.*

Proof. Choose a vertex v with minimal coefficient $E(v)$. Set $m := E(v)+1$, and suppose that $|E - m\,v|$ is nonempty. Then there exists an effective divisor, E', such that $E - m\,v \sim E'$, or equivalently $E \sim E' + m\,v$. But by proposition 3.11, this linear equivalence must be an equality: $E = E' + m\,v = E' + (E(v)+1)\,v$. But this is a contradiction since v appears on the right hand side at least once more than on the left. It follows that $|E - m\,v| = \emptyset$, so $r(E) \leq E(v)$. On the other hand, the minimality of $E(v)$ implies that if F is any effective divisor of degree less than or equal to $E(v)$, then $E - F$ is effective. It follows that $r(E) = E(v)$ as claimed. $\square$

It might seem that the rank function, $r(D)$, is even more difficult to compute than the size of the complete linear system $|D|$. Indeed, to make a straightforward computation of $r(D)$, one would need to answer the entire sequence of questions:

Is $|D|$ nonempty?
  If so, then are all $|D - v_i|$ nonempty?
    If so, then are all $|D - v_i - v_j|$ nonempty?
      If so, then what about all $|D - v_i - v_j - v_k|$?
        etc.,

each of which involves the investigation of lattice points in a convex polytope. In fact, the problem of computing the rank of a general divisor on a general graph is **NP**-hard ([9]), which means it is likely