Problems for Chapter 2

2.1. Let $\widetilde{L} := \widetilde{L}_q$ and $\widetilde{L}' := \widetilde{L}_{q'}$ be the reduced Laplacians of a graph G with respect to vertices q and q', respectively. Exhibit an explicit isomorphism $\operatorname{cok}(\widetilde{L}) \simeq \operatorname{cok}(\widetilde{L}')$ stemming from Proposition 2.11.

2.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs, and pick vertices $v_1 \in V_1$ and $v_2 \in V_2$. Let G be the graph obtained from G_1 and G_2 by identifying vertices v_1 and v_2, thus gluing the two graphs together at a vertex.

(a) Prove that $\operatorname{Jac}(G) \simeq \operatorname{Jac}(G_1) \times \operatorname{Jac}(G_2)$.

(b) Prove that every finite abelian group is the Jacobian of some graph.

2.3. Provide details for the proof of Proposition 2.24 by showing the dashed mapping in commutative diagram 2.2 is well-defined and bijective.

2.4. The uniqueness statement of Theorem 2.19 comes in two versions: case 1 and case 2. Use the Chinese remainder theorem to show the cases are equivalent.

2.5. The *house graph*, H, is displayed in Figure 3. Determine the

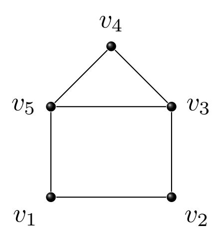

Figure 3. The house graph.

structure of $\operatorname{Jac}(H)$ by computing the Smith normal form for the reduced Laplacian of H.

2.6. Compute the Smith normal form for the reduced Laplacian of the complete graph, K_n. (Hint: start the computation by adding columns 2 through $n-1$ to column 1.) Conclude that

$$\operatorname{Jac}(K_n) \simeq \mathbb{Z}_n^{n-2}.$$

Problems for Chapter 3

3.1. Is the an algorithm for the dollar game based on benevolence as follows? As long as some vertex is in debt, a vertex that is not in debt lends. Repeat until everyone is out of debt or until a state is reached at which continuing would force each vertex to have made a lending move, at which point the game is certifiably unwinnable. Prove or disprove.

3.2. Let C_4 be the cycle graph with vertices v_1, v_2, v_3, v_4 listed in order around the cycle. Let $D = -2v_1 - 2v_2 + 2v_3 + 3v_4 \in \mathrm{Div}(C_4)$.

(a) Find the firing script, σ, determined by the greedy algorithm for the dollar game.

(b) Let L be the Laplacian for C_4. Verify that $D - L\sigma$ is effective.

3.3. Let C_n be the cycle graph with vertices $v_1, \ldots, v_n$, listed in order around the cycle. Suppose $n = 2m$ with $m > 1$, and let $D = -2v_1 + 2v_{m+1} \in \mathrm{Div}(C_n)$. What is the firing script produced by the greedy algorithm for the dollar game applied to D?

3.4. Let G be the house graph picture in Figure 3 in the Problems for Chapter 2. Let $q = v_1$, and compute linearly equivalent q-reduced divisors for the following divisors on G:

(a) $D_1 = (-3, 2, 4, -2, 1)$

(b) $D_2 = (2, 1, -5, 2, 2)$

(c) $D_3 = (0, -2, -2, 0, 0)$.

3.5. Consider a variation of the protocol for parking cars described in §3.4.1. There are still n cars, $C_1, \ldots, C_n$, but this time there is one extra parking space, numbered $n+1$, and the spaces are arranged in a circle. Car C_i prefers to park in space $p_i \in \{1, \ldots, n+1\}$. Other than that, the rules are essentially the same: each car in turn drives to its preferred spot and parks there if possible. Otherwise, it drives on to the next available spot. Since the spaces are arranged in a circle, each car will eventually park. Call these preference lists *circular parking functions*.

Credit Matthias Beck with this terminology. Is it his?

(a) After the cars park according to a given circular parking function, there is one empty parking space. Show that the number of