Problems for Chapter 2

2.1. Suppose we reorder the vertices of G by applying a permutation $\pi \in S_n$ to the ordering $\{v_1, v_2, \ldots, v_n\}$. How does this affect the Laplacian matrix?

Too trivial?

2.2. Let $\widetilde{L} := \widetilde{L}_q$ and $\widetilde{L}' := \widetilde{L}_{q'}$ be the reduced Laplacians of a graph G with respect to vertices q and q', respectively. Describe an isomorphism $\operatorname{cok}(\widetilde{L}) \simeq \operatorname{cok}(\widetilde{L}')$.

2.3.* Characterize all graphs with Picard group isomorphic to $\mathbb{Z}_4$. (Do no count graphs obtained from others by attaching a tree at a vertex. In other words, only graphs with no vertices of degree 1.)

2.4. Provide details for the proof of Proposition 2.24 by showing the dashed mapping in commutative diagram 2.2 is well-defined and bijective.

2.5. The uniqueness statement of Theorem 2.19 comes in two versions: case 1 and case 2. Use the Chinese remainder theorem to show the cases are equivalent.

2.6. The *house graph*, H, is displayed in Figure 3. Determine the

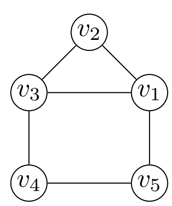

Figure 3. The house graph.

structure of $\operatorname{Jac}(H)$ by computing the Smith normal form for the reduced Laplacian of H.

2.7. Compute the Smith normal form for the reduced Laplacian of the complete graph, K_n. (Hint: start the computation by adding columns 2 through n to column 1.) Conclude that

$$\operatorname{Jac}(G) \simeq \mathbb{Z}_n^{n-2}.$$

2.8. Let C_n denote the cycle graph with n vertices (n vertices placed on a circle). Use the computation of the Smith normal form of the reduced Laplacian for C_n to give an explicit isomorphism between $\mathrm{Jac}(G)$ and $\mathbb{Z}_n$.

2.9. Let K_n^- be the graph obtained by deleting one edge from the complete graph on n vertices. Find the structure of $\mathrm{Jac}(K_n^-)$ by computing the Smith normal form of the reduced Laplacian.

Problems for Chapter 3

3.1. Is there an algorithm for the dollar game based on benevolence? As long as some vertex is in debt, a vertex that is not in debt lends. Repeat until everyone is out of debt or until a state is reached at which continuing would force each vertex to have made a lending move, at which point the game is certifiably unwinnable. Prove or disprove.

3.2. Consider a variation of the protocol for parking cars described in §3.4.1. There are still n cars, $C_1, \ldots, C_n$, but this time there is one extra parking space, numbered $n+1$, and the spaces are arranged in a circle. Car C_i prefers to park in space $p_i \in \{1, \ldots, n+1\}$. Other than that, the rules are essentially the same: each car in turn drives to its preferred spot and parks there if possible. Otherwise, it drives on to the next available spot. Since the spaces are arranged in a circle, each car will eventually park. Call these preference lists *circular parking functions*.

Credit Matthias Beck with this terminology. Is it his?

(a) After the cars park according to a given circular parking function, there is one empty parking space. Show that the number of circular parking functions that leave space i empty is the same as the number that leave space 1 empty, for each i.

(b) Show that a circular parking function is an actual parking functions if and only if it leaves space $n+1$ empty.

(c) Conclude that the number of ordinary parking functions of length n is $(n+1)^{n-1}$.

3.3. Let $D \in \mathrm{Div}(G)$, and fix a source vertex $q \in V$. The proof of the existence and uniqueness theorem for q-reduced divisors guarantees the existence of a firing script σ such that $D \xrightarrow{\sigma} D'$ where $D'(v) \geq 0$ for all $v \neq q$. Using this fact, mimic the proof of the validity of Algorithm 1, Chapter 3, to verify that the following greedy algorithm brings the non-source vertices of D out of debt:

3.4. Let $c \in \mathrm{Config}(G, q)$. Fixing an ordering of the vertices, let $\widetilde{L}$ be the reduced Laplacian of G and identify c with an integer vector in $\mathbb{Z}^{n-1}$, as usual. Let

$$\sigma := \lfloor \widetilde{L}^{-1} c \rfloor \in \mathbb{Z}^{n-1}$$