Problems for Chapter 1

1.1. Generalize the argument in Example 1.17 to show that if $D = 2\,v_1 + v_2 - 2\,v_3$ on the triangle graph C_3, then

$$[D] = \{(1 + 3k + m)\,v_1 - (3k + 2m)\,v_2 + m\,v_3 : k, m \in \mathbb{Z}\}.$$

1.2. Let C_n denote the cycle graph with vertices $v_0, \ldots, v_{n-1}$ arranged counter-clockwise in a circle. Define divisors $D_0 = 0$ and $D_i = v_i - v_0$ for $i = 1, \ldots, n - 1$.

(a) Directly from set-lendings, show by induction that $k\,[D_1] = [D_k]$ for $k = 0, 1, \ldots, n - 1$ and that $n\,[D_1] = 0$. Thus, if we take indices modulo n, we have $k\,[D_1] = [D_k]$ for all $k \in \mathbb{Z}$.

(b) Consider the homomorphism $\psi\colon \mathbb{Z}_n \to \mathrm{Jac}(C_n)$ given by $k \mapsto k\,[D_1]$ (which is well-defined by part (a). Show that $\mathrm{Jac}(C_n)$ is isomorphic to $\mathbb{Z}_n$ by exhibiting the inverse.

(c) Find, without proof, the complete linear system of the divisor $D = -2\,v_0 + v_1 + 2\,v_2 + v_3$ on C_4.

1.3. Let P_n be the *path graph* on n vertices consisting of $n - 1$ edges connected to form a line as in Figure 5.

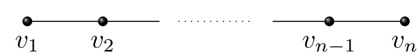

Figure 5. The path graph of length n.

(a) Given a divisor $D = \sum_{i=1}^{n} a_i v_i \in \mathrm{Pic}(P_n)$, describe a sequence of set-lendings (and borrowings) which show $D \sim \deg(D)\,v_n$.

(b) Prove $\mathrm{Pic}(P_n) \simeq \mathbb{Z}$.

(c) For more of challenge, show that the Picard group of every tree is isomorphic to $\mathbb{Z}$.

1.4. Prove Proposition 1.19.