Math 372

Dirichlet Series

Notation: $f \overset{D}{\longleftrightarrow} \{a_n\}$ if $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \cdots$ and $s \in \mathbb{C}$.

Multiplicative functions

$f: \mathbb{Z}_{>0} \to \mathbb{C}$ is multiplicative if $f(mn) = f(m)f(n)$ whenever $\gcd(m,n) = 1$.

Thus, if f is multiplicative and $f \not\equiv 0$, then $f(1) = 1$.

(If $f(1) = 0$, then $f(m) = f(1 \cdot m) = f(1)f(m) = 0 \quad \forall m \in \mathbb{N}$.

Otherwise, i.e., when $f(1) \neq 0$, we have $f(1) = f(1 \cdot 1) = f(1)f(1) \Rightarrow f(1) = 1$.)

Example $f(120) = f(2^3 \cdot 3 \cdot 5) = f(2^3) f(3) f(5)$.

Note: Multiplicative functions are determined by their values at powers of primes.

Examples

- $f(n) = 1 \quad \forall n$
- $d(n) = \#$ divisors of n
- $\phi(n) =$ Euler ϕ-function
  $= \#\{m \in \mathbb{Z} : 1 \leq m \leq n \text{ and } \gcd(m,n) = 1\}$.

For each multiplicative function, f, there is a corresponding Dirichlet series:

$$\sum_{n \geq 1} \frac{f(n)}{n^s} \overset{D}{\leftrightsquigarrow} \{f(n)\}.$$

Example If $f \equiv 1$, then we get $\zeta(s)$.

Theorem. If f is multiplicative, then

$$\sum_{n\geq 1} \frac{f(n)}{n^s} = \prod_{\substack{p \\ p \text{ prime}}} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + \cdots\right).$$

Pf/ What is the coefficient of $\frac{1}{n^s}$ on the RHS? Say $n = \prod_i p_i^{e_i}$ is the factorization of n into primes. The coefficient is then $\prod_i f(p^{e_i}) = f(n)$, since f is multiplicative. □

Example Riemann zeta: $\zeta(s) = \sum_n \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 + p^{-s} + p^{-2s} + \cdots\right)$

$= \prod_p \frac{1}{1-p^{-s}}$ (Since $\zeta(1) = \sum \frac{1}{n} = \infty$, this gives a proof that there are infinitely many primes.)

Example $\frac{1}{\zeta(s)} = \prod_p (1-p^{-s}) = \sum^{\uparrow} \frac{\mu(n)}{n^s}$

Apply Theorem in reverse to the multiplicative function μ.

where $\mu : \mathbb{Z}_{>0} \to \{-1, 0, 1\}$ is the Möbius function

defined as the multiplicative function with $\mu(p^{e_i}) = \begin{cases} 1 & \text{if } e_i = 0 \\ -1 & \text{if } e_i = 1 \\ 0 & \text{if } e_i > 1 \end{cases}$.

For example, $\mu(1) = 1$, $\mu(2 \cdot 3 \cdot 5) = (-1)^3 = -1$, $\mu(2 \cdot 3^2 \cdot 5) = 0$.

Möbius inversion

Suppose given sequences $\{a_n\}, \{b_n\}$ in $\mathbb{C}$ with

$$a_n = \sum_{d \mid n}^{\uparrow} b_d .$$

Consider the corresponding Dirichlet series: $A \overset{D}{\longleftrightarrow} \{a_n\}$, $B \overset{D}{\longleftrightarrow} \{b_n\}$.

Thm $A \overset{D}{\leftrightsquigarrow} \{a_n\} = \{\sum_{d|n} b_d\} = \{\sum_{d|n} b_d \cdot 1\} \overset{D}{\leftrightsquigarrow} B\zeta$

$\Rightarrow B = A\frac{1}{\zeta} \overset{D}{\leftrightsquigarrow} \{\sum_{d|n} a_d \mu(\frac{n}{d})\}.$

We've proved the following

Thm. If $a_n = \sum_{d|n} b_d \quad \forall n$, then $b_n = \sum_{d|n} a_d \mu(\frac{n}{d})$, where μ is the Möbius function.

Example Say a bit string is primitive if it is not the concatenation of identical smaller-length strings.

Not primitive

101|101|101

1111

0000

101101101

primitive

10

11110

10110

Let $f(n)$ = # primitive bit strings of length n.

Then

$$2^n = \begin{array}{c}\text{total number of bit} \\ \text{strings of length } n\end{array} = \sum_{d \mid n} f(d)$$

By Möbius inversion, $f(n) = \sum_{d \mid n} 2^d \mu\left(\frac{n}{d}\right)$.

For example, $f(4) = 2\mu(4) + 2^2\mu(2) + 2^4\mu(1)$

$= 0 - 4 + 16 = 12$

So 12 of the 16 bit strings of length 4 are primitive, leaving 4 non-primitive bit strings:

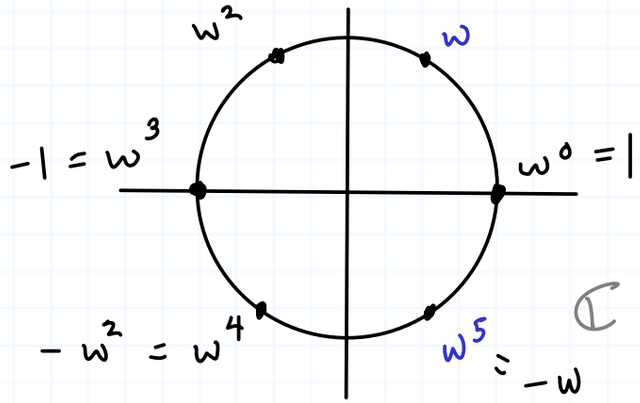
0000, 0101, 1010, 1111

Example Cyclotomic polynomials

n^{th} roots of $1 = (\text{solutions in } \mathbb{C} \text{ to } x^n = 1) = \{ e^{2\pi i k/n} : k = 0, 1, \ldots, n-1 \}$

primitive n^{th} roots of $1 = \{ e^{2\pi i k/n} : k = 0, \ldots, n-1 \text{ and } \gcd(k,d) = 1 \}$.

Example $n = 6$. Let $w = e^{2\pi i/6}$. The 6th roots of 1 are



blue = primitive roots

If w is primitive then each root of 1 is some power of w.

Facts: n^{th} roots of $1 = \bigcup_{d|n} \{ \text{primitive } d^{\text{th}} \text{ roots of } 1 \}$ (Check w/ the above example.)

$$x^n - 1 = \prod_{k=0}^{n-1} (x - e^{2\pi i k/n})$$

Def. The nth cyclotomic polynomial is $\Phi_n(x) := \prod_{\substack{k=0,\ldots,n-1 \\ \gcd(k,n)=1}} (x - e^{2\pi i k/n})$.

Then by the Facts,

$$x^n - 1 \overset{\star}{=} \prod_{d \mid n} \Phi_d(x)$$

$$\Rightarrow \log(x^n - 1) = \sum_{d \mid n} \log \Phi_d(x)$$

Möbius inversion

$$\Rightarrow \log \Phi_n(x) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \log(x^d - 1)$$

$$\Rightarrow \Phi_n(x) = \prod_{d \mid n} (x^d - 1)^{\mu\left(\frac{n}{d}\right)}$$

Examples

$$\Phi_1(x) = x - 1$$

$$\Phi_2(x) = (x-1)^{\mu(2)} (x^2-1)^{\mu(1)} = \frac{x^2-1}{x-1} = x + 1$$

$$\Phi_3(x) = (x-1)^{\mu(3)} (x^3-1)^{\mu(1)} = \frac{x^3-1}{x-1} = x^2 + x + 1$$

$$\Phi_4(x) = (x-1)^{\mu(4)} (x^2-1)^{\mu(2)} (x^4-1)^{\mu(1)} = \frac{x^4-1}{x^2-1} = x^2 + 1$$

$$\Phi_5(x) = (x-1)^{\mu(5)}(x^5-1)^{\mu(1)} = \frac{x^5-1}{x-1} = x^4+x^3+x^2+x+1$$

$$\Phi_6(x) = (x-1)^{\mu(6)}(x^2-1)^{\mu(3)}(x^3-1)^{\mu(2)}(x^6-1)^{\mu(1)} = \frac{(x-1)(x^6-1)}{(x^2-1)(x^3-1)} = \frac{x^3+1}{x+1} = x^2-x+1.$$

So $x^6-1 = \prod_{d|6} \Phi_d(x) = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$.

It turns out that each Φ_n is irreducible over $\mathbb{Q}$, i.e., can't be factored into lower degree polynomials with coefficients in $\mathbb{Q}$.