Math 372 Homework for Wednesday, Week 4

Problem 1. Let H be a subgroup of $\mathfrak{S}_n$ acting on $2^{[n]}$ as in class. If $\mathcal{O}$ and $\mathcal{O}'$ are two orbits of this action, define $\mathcal{O} \leq \mathcal{O}'$ if there exists $x \in \mathcal{O}$ and $y \in \mathcal{O}'$ such that $x \leq y$, i.e., x is a subset of y. Prove that $\leq$ is a partial ordering on the set of orbits (thus making B_n/H into a poset).

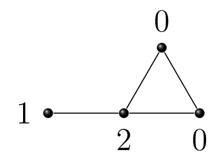
Problem 2. Let $G = \{\iota, \pi\}$ be a group of order two (with identity element ι). Let G act on $\{1, 2, 3, 4\}$ by $\pi \cdot 1 = 2$, $\pi \cdot 2 = 1$, $\pi \cdot 3 = 3$, $\pi \cdot 4 = 4$. Draw the Hasse diagram for the quotient poset B_4/G.

Problem 3. A $(0, 1)$-necklace of length n and weight i is a circular arrangement of i 1's and $n - i$ 0's. For instance, the $(0, 1)$-necklaces of length 6 and weight 3 are (writing a circular arrangement linearly) 000111, 001011, 010011, and 010101. (Cyclic shifts of a linear word represent the same necklace, e.g., 000111 is the same as 110001.) Let N_n denote the set of all $(0, 1)$-necklaces of length n. Define a partial order on N_n by letting $u \leq v$ if we can obtain v from u by changing some 0's to 1's. It's easy to see (you may assume it) that N_n is graded of rank n, with the rank of a necklace being its weight.

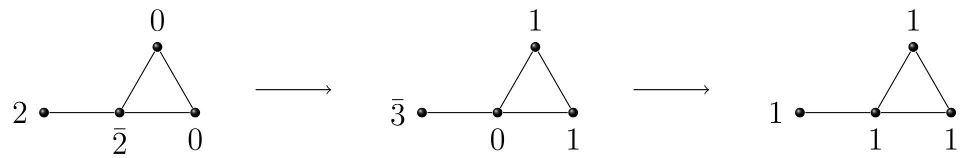
Use Theorem 5.8 to show that N_n is rank-symmetric, rank-unimodal, and Sperner.

Problem 4. In Example 5.4 (b) in our text contains a drawing of the Hasse diagram of B_5/G, where G is generated by the cycle $(1, 2, 3, 4, 5)$. Using the vertex labels used in that drawing, compute explicitly $\widehat{U}_2(\mathcal{O}_{12})$ and $\widehat{U}_2(\mathcal{O}_{13})$ as linear combinations of 123 and 124, where $\widehat{U}_2$ is defined as in the proof of Theorem 5.8. What is the matrix of $\widehat{U}_2$ with respect to the standard (lexicographically ordered) bases for $(B_5/G)_2$ and $(B_5/G)_3$?

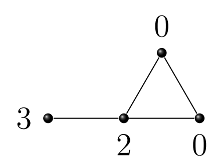
Problem 5. (An unusual chip-firing game.) Let $G = (V, E)$ be a graph. A *chip configuration* on G is the assignment of a nonnegative integer to each vertex $v \in V$. Think of the integer as specifying a number of *chips* to place on each vertex. Say a vertex is *unstable* if it has 2 or more chips. In that case, we can *fire* the vertex by removing 2 chips from it and adding 1 chip to each of its neighbors (vertices connected to it by an edge). A chip configuration is *stable* if it has no unstable vertices. Sometimes, by consecutively firing a finite list of vertices, a chip configuration can be transformed into a stable configuration, in which case we say the configuration is *stabilizable*. For instance, the following configuration is stabilizable (integers above vertices represent numbers of chips):



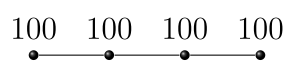
Here is a stabilization (with the vertex to be fired indicated by putting a bar above its number of chips):



One may check that the following configuration is unstabilizable:



On the other hand, *every* configuration on a path graph (vertices connected by a line of edges) is stabilizable. For instance, on scratch paper, show that the following configuration for the path graph with four vertices, P_4, is stabilizable:



For this problem, find a graph G that is not a path graph having the property that every configuration on G is stabilizable. [For more of a challenge, find an infinite family of such graphs. For even more of a challenge, classify all such graphs.]