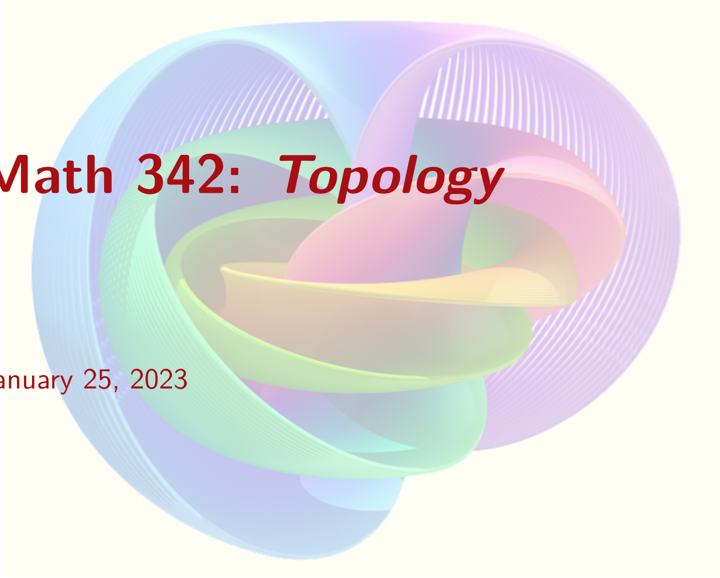

Math 342: *Topology*

January 25, 2023

Today

- Bases
- Examples
- Subbases

Bases

Recall definitions:

A *basis* for a topology on a set X is a collection $\mathcal{B}$ of subsets of X satisfying

1. For each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, i.e., the basis elements *cover* X:
$$\cup_{B \in \mathcal{B}} B = X.$$
2. If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The *topology generated by* a basis $\mathcal{B}$ is

$$\begin{aligned}\mathcal{T} &= \{U \subseteq X : \text{for all } x \in U, \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \subseteq U\} \\ &= \{U \subseteq X : U \text{ is a union of elements of } B\},\end{aligned}$$

Bases

Lemma. Suppose that X is a topological space and $\mathcal{C}$ is a collection of open sets in X. Suppose that for each open set U in X and for each $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology on X.

Proof. We need to prove that $\mathcal{C}$ is basis and that the topology it generates is the given topology on X. We first check the two properties in the definition of a basis.

- Why does $\mathcal{C}$ cover X?
- Next, suppose $x \in C_1 \cap C_2$ for some $C_1, C_2 \in \mathcal{C}$. Why does there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$.

Now let $\mathcal{T}$ be the original topology on X, and let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$. How do we show $\mathcal{T} = \mathcal{T}'$?

Fineness and coarseness

Definition. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set X. If $\mathcal{T}' \supseteq \mathcal{T}$, then $\mathcal{T}'$ is *finer* than $\mathcal{T}$ and $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

Lemma. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. The following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and each $B \in \mathcal{B}$ with $x \in B$, there exists $B' \in \mathcal{B}'$ with $x \in B' \subseteq B$.

Proof. To appear after some examples.

Bases for standard topology on $\mathbb{R}^n$

Bases for topologies on $\mathbb{R}^n$:

- $\mathcal{B}$ = collection of open balls:

$$B(r,x) = \{y \in \mathbb{R}^n : |x-y| < r\}.$$

- $\mathcal{B}'$ = collection of open rectangles:

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n).$$

Exercise: check these are bases.

Claim: the topologies generated by these bases are the same.
Check by using the Lemma.

Lower limit topologies

Let $\mathcal{B}$ the collection of intervals of $\mathbb{R}$ of the form $[a, b)$ where $a < b$.

Claim: $\mathcal{B}$ is a basis.

Definition. The topology generated by $\mathcal{B}$ is called the *lower limit topology* on $\mathbb{R}$.

Claim: The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$. Check using the lemma.

For instance, why is $(0, 1)$ open in the lower limit topology?

Answer: $(0, 1) = \cup_{a>0}[a, 1)$, and $\cup_{a>0}[a, 1)$ is a union of basis elements for the lower limit topology.

Fineness and coarseness

Lemma. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. The following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$ (i.e., $\mathcal{T}' \supseteq \mathcal{T}$).
2. For each $x \in X$ and each $B \in \mathcal{B}$ with $x \in B$, there exists $B' \in \mathcal{B}'$ with $x \in B' \subseteq B$.

Proof. ($1 \Rightarrow 2$) Let $x \in B \in \mathcal{B}$. Then $B \in \mathcal{T} \subseteq \mathcal{T}'$. Since $x \in B \in \mathcal{T}'$, by definition of $\mathcal{T}'$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

($2 \Rightarrow 1$) Let $U \in \mathcal{T}$. By definition of $\mathcal{T}$, for each $x \in U$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \in \mathcal{B}$. By 2, there exists $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subseteq B_x$. Then $U = \cup_{x \in U} B'_x \in \mathcal{T}'$.

Subbases

Definition. A *subbasis* $\mathcal{S}$ for a topology on a set X is a collection of subsets of X that cover X. The *topology generated by* $\mathcal{S}$ is the set of arbitrary unions of finite intersections of elements of $\mathcal{S}$.

Claim: The topology generated by a subbasis $\mathcal{S}$ is a topology.

Proof. Let $\mathcal{B} = \{$all finite intersections of elements of $\mathcal{S}\}$. To prove the claim, it suffices to show that $\mathcal{B}$ is a basis. Why does it cover X? Next, given $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, why does there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. (Answer: let $B_3 = B_1 \cap B_2$. Then $B_3 \in \mathcal{B}$?)