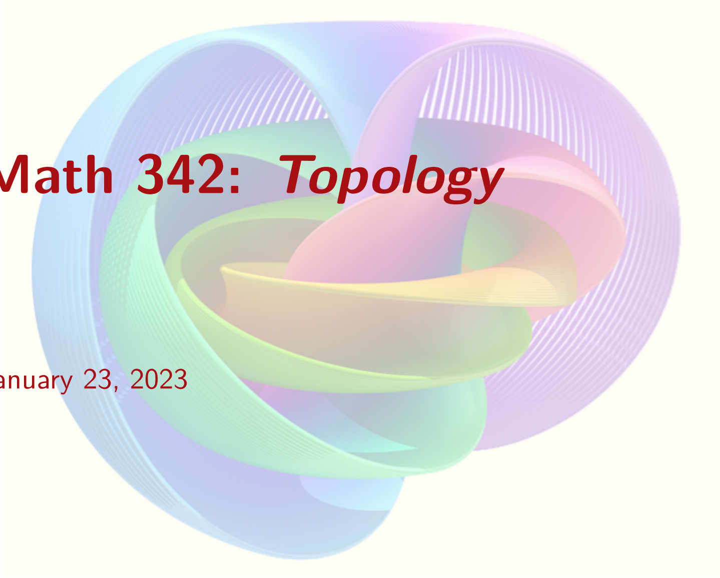
Math 342: ***Topology***



January 23, 2023

Today

- Definition of a topological space.
- Basis for a topology.

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A subset $C \subseteq X$ is *closed* if $X \setminus C$ (complement) is open.

The standard topology on $\mathbb{R}$

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Equivalently (exercise), we can declare a subset open if it is a union of open intervals.

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  Solution. Yes. (Check the axioms.)

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Remark. A given set can have several different topologies.

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   Solution. False. For instance, the interval $[0, 1)$ is neither open nor closed in the standard topology for $\mathbb{R}$.

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Example: $\mathbb{R}^n$**.** The collection of open balls in $\mathbb{R}^n$ is a basis for the usual topology on $\mathbb{R}^n$. We will see that collection of open cubes is also a basis for the same topology.

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Proposition. Let $\mathcal{B}$ be a basis for the topology on a set X, and let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X.

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The *topology generated by* a basis $\mathcal{B}$ is

$$\begin{aligned} \mathcal{T} &= \{U \subseteq X : \text{for all } x \in U, \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \subseteq U\} \\ &= \{U \subseteq X : U \text{ is a union of elements of } \mathcal{B}\}, \end{aligned}$$

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2. $X \in \mathcal{T}$ follows from property 1.

3. That $\mathcal{T}$ is closed under unions is easily seen from our second characterization of $\mathcal{T}$ as the collection of subsets that are unions of elements of $\mathcal{B}$.

4. By induction, to show that $\mathcal{T}$ is closed under finite unions, it suffices to show that if U_1 and U_2 are in $\mathcal{T}$, then so is $U_1 \cap U_2$. To prove this, we use the first characterization of $\mathcal{T}$. Suppose $U_1, U_2 \in \mathcal{T}$, and let $x \in U_1 \cap U_2$. Then there exist B_1 and B_2 in $\mathcal{T}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. So $x \in B_1 \cap B_2$. Using property 2 of the definition of a basis, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus, $U_1 \cap U_2 \in \mathcal{T}$.