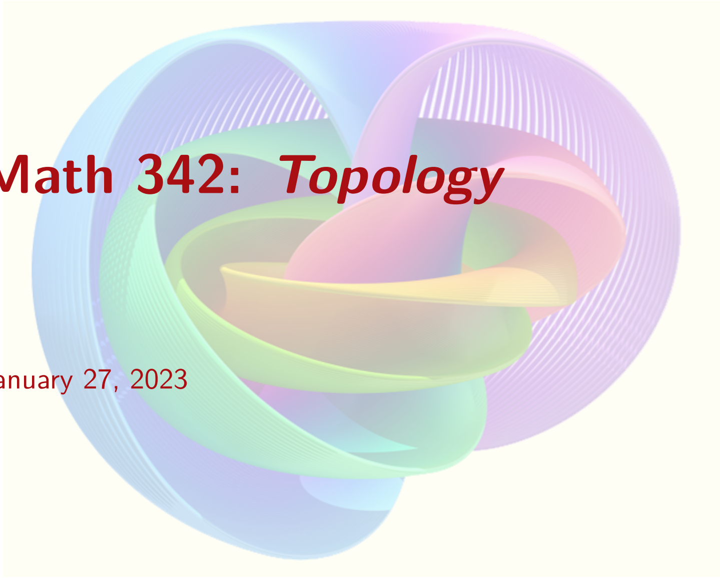
Math 342: *Topology*



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Today

- Definition of a topology in terms of closed sets.
- Neighborhoods.
- Limit points, interior, and closure of a set.

Characterization of a topology using closed sets

Recall that if X is a topological space and $C \subseteq X$, then C is *closed* if $X \setminus C$ is open.

Theorem. Let X be a topological space. Then

1. $\emptyset$ and X are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of close sets are closed.

Proof. The result follows from De Morgan's laws:

$$X \setminus (\cap_{\alpha \in J} C_\alpha) = \cup_{\alpha \in J} (X \setminus C_\alpha)$$

$$X \setminus (\cup_{i=1}^{n} C_\alpha) = \cap_{i=1}^{n} (X \setminus C_\alpha) . \qquad \square$$

Remark. We could define a topology by specifying a collection satisfying 1, 2, and 3, and calling those the closed sets.

Neigborhoods and limit points

Let X be a topological space. A *neighborhood* of $x \in X$ is an open subset containing containing x.

The point $x \in X$ is a *limit point* of $A \subseteq X$ if every neighborhood U of x satisfies $(U \setminus \{x\}) \cap A \neq \emptyset$.

Remarks.

1. This definition captures the idea of x being "arbitrarily close to points in A".
2. x may or may not be in A.
3. Points in A are not necessarily limit points.

Exercises

1. In $\mathbb{R}^n$, x is a limit point of $A \subseteq \mathbb{R}^n$ if and only if for all $\varepsilon > 0$, we have $(B(x, \varepsilon) \setminus \{x\}) \cap A \neq 0$. Why?
2. What are the limit points of $(0, 1] \subset \mathbb{R}$?
3. What are the limit points of $\{1/n : n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$?

Interior and closure

Definition. Let X be a topological space, and let $A \subseteq X$. The *interior* of A, denoted $\mathring{A}$ or $\mathrm{Int}(A)$, is the union of all open sets contained in A.

The *closure* of A, denoted $\overline{A}$, is the intersection of all closed sets containing A.

Remarks.

1. $\mathring{A}$ and $\overline{A}$ always exist.
2. $\mathring{A}$ is open in X, and $\overline{A}$ is closed.
3. $\mathring{A} \subseteq A \subseteq \overline{A}$.
4. $\mathring{A}$ is the largest open set contained in A, and $\overline{A}$ is the smallest closed set containing A.
5. A is open if and only if $A = \mathring{A}$.
6. A is closed if and only if $A = \overline{A}$.

Interior and closure

Proposition. Let $x \in A$. Then $x \in \mathring{A}$ if and only if there exists a neighborhood of x contained in A.

Proof. If $x \in \mathring{A}$, then $\mathring{A}$, itself, is a neighborhood of x contained in A. Conversely, if U is a neighborhood of x contained in A, then $x \in U \subseteq \mathring{A}$. $\square$

Interior and closure

Theorem. Let X be a space and let $A \subseteq X$. The following are equivalent:

1. $x \in \overline{A}$,
2. Every neighborhood of x intersects A.
3. Given a basis $\mathcal{B}$ for the topology on X, every basis element that contains x intersects A.

Proof. ($1 \Rightarrow 2$) We prove the contrapositive. Let U be a neighborhood of x that does not intersect A. Define $C = X \setminus U$. Then C is a closed set containing A. Thus,

$$x \notin C \supseteq \overline{A}.$$

($2 \Rightarrow 3$) Every basis element containing x is a neighborhood of x.

($3 \Rightarrow 1$) Proof continued on next page.

Interior and closure

Theorem. Let X be a space and let $A \subseteq X$. The following are equivalent:

1. $x \in \overline{A}$,
2. Every neighborhood of x intersects A.
3. Given a basis $\mathcal{B}$ for the topology on X, every basis element that contains x intersects A.

Proof continued.
($3 \Rightarrow 1$) We prove the contrapositive. Suppose $x \notin \overline{A}$. This means that there exists a neighborhood U of x such that $(U \setminus \{x\}) \cap A = \emptyset$. By definition of the topology generated by a basis, there exists a basis element B such that $x \in B \subseteq U$. Then by set theory,

$$(B \setminus \{x\}) \cap A \subseteq (U \setminus \{x\}) \cap A = \emptyset.$$

$\square$

Interior and closure

Find the interior and closure of each of the following subsets of $\mathbb{R}$ (with the standard topology):

1. $A = (-2, -1) \cup (-1, 3] \cup [4, 5] \cup \{6\}$.

   Solution: We have

   $$\mathring{A} = (-2, -1) \cup (-1, 3) \cup (4, 5) \quad \text{and} \quad \overline{A} = [-2, 3] \cup [4, 5] \cup \{6\}.$$

2. $B = \{1/n : n \in \mathbb{Z}\}$.

   Solution:

   $$\mathring{B} = \emptyset \quad \text{and} \quad \overline{B} = ?$$

   The next theorem will help us determine $\overline{B}$.

Closure and limit points

Theorem. Let X be a topological space, and let $A \subseteq X$. Let A' be the set of limit points of A. Then $\overline{A} = A \cup A'$.

Proof. We first show $A \cup A' \subseteq \overline{A}$. If $x \in A$, then $x \in \overline{A}$. If $x \in A'$, then every neighborhood of x meets A. By our previous Theorem, we conclude that $x \in \overline{A}$. We are done with this inclusion.

Next, we show $\overline{A} \subseteq A \cup A'$. Suppose $x \in \overline{A}$ and $x \notin A$. To show $x \in A'$, let U be a neighborhood of x. By the previous Theorem, we know $U \cap A \neq \emptyset$. Since $x \notin A$, it follows that $(U \setminus \{x\}) \cap A \neq \emptyset$. Hence $x \in A'$. Therefore, $\overline{A} = A \cup A'$. $\square$

Corollary. A subset is closed if and only if it contains its limit points.

Example

What is the closure of the set $B = \{1/n : n \in \mathbb{Z}\}$?

The only limit point of B is 0. Hence,

$$\overline{B} = B \cup \{0\}.$$