Math 332, Tuesday, Feb. 24

* Solutions to HW are now posted.

* Reading + Quiz for Thursday: Know the definitions and statements of the main results in sections 18, 19, and 20.

<u>Today</u> Group actions: orbits and stabilizers.

$x \in X$, $\quad \mathrm{Stab}(x) := \{g \in G : gx = x\} < G$, Check!

$\mathrm{Orb}(x) := Gx = \{gx : g \in G\} \subseteq X$

$g \in G \quad \mathrm{Fix}(g) = \{x \in X : gx = x\} \subseteq X$

Thm (Orbit-stabilizer thm) If G is finite then for each $x \in X$,

$$|G| = |\mathrm{Orb}(x)|\,|\mathrm{Stab}(x)|$$

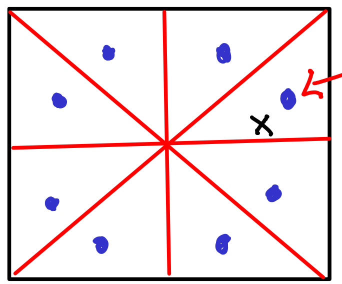
Thm (Burnside's Lemma) If G and X are finite, then

$$\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

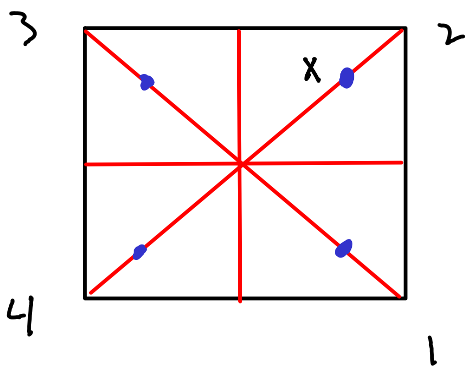
The right hand side is the average number of fixed points of an element of G

Examples

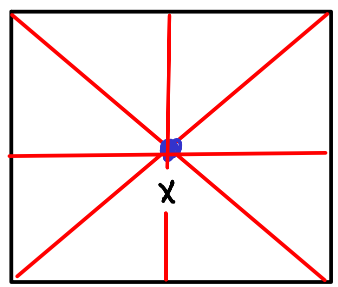
(1) Orbits of points under D_4 action on the plane.



The orbit of this pt. : $|\mathrm{orb}(x)|\,|\mathrm{stab}(x)| = 8 \cdot 1 = |D_4|$



$$|\mathrm{orb}(x)|\,|\underbrace{\mathrm{stab}(x)}_{\{(),\,(13)\}}| = 4 \cdot 2 = |D_4|$$



$$|\mathrm{orb}(x)|\,|\mathrm{stab}(x)| = 1 \cdot 8 = |D_4|$$

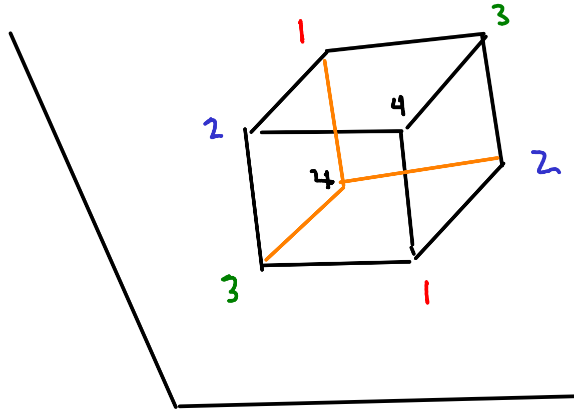
(2) How big is the group of rotational symmetries of the cube?

Solution / The 3-D rotations act on the faces of the cube.

Fix a face, F. Then

$$|G| = |\text{orb}(F)|\,|\text{stab}(F)| = 6 \cdot 4 = 24$$

So G is a group of order 24. [To see that, in fact, $G \cong S_4$ consider G acting on the 4 diagonals of the cube.



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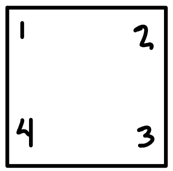
(3) D_4 acting on the vertices of a square:

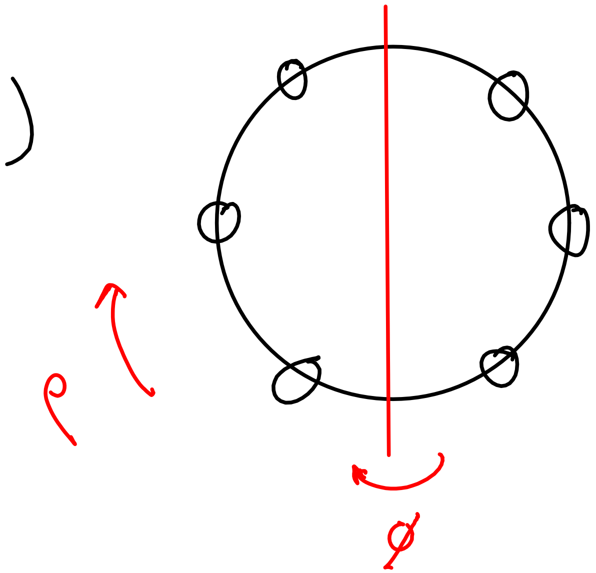


g	1	$\overset{(1234)}{\rho}$	ρ^2	ρ^3	$\overset{(13)}{\phi}$	$\rho\phi$	$\rho^2\phi$	$\rho^3\phi$
$\lvert\text{Fix}(g)\rvert$	4	0	0	0	2	0	2	0

$$\#\text{orbits} = \frac{1}{8}(4+0+0+0+2+0+2+0) = \frac{1}{8}\cdot 8 = 1$$

Note: $\text{orb}(i) = \{1,2,3,4\}$ for each i.

(4)



How many 6-bead necklaces with bead selected from 3 colors B R O ?

Solution: Basic designs, counting symmetrical designs as different: 3^6

But, D_6 acts on this set of 3^6 colorings. We would like to count the colorings that are distinct modulo these symmetries, i.e., the number of orbits.

g	1	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ϕ	$\rho\phi$	$\rho^2\phi$	$\rho^3\phi$	$\rho^4\phi$	$\rho^5\phi$
$\lvert Fix(g) \rvert$	3^6	3	3^2	3^3	3^2	3	3^3	3^4	3^3	3^4	3^3	3^4

Burnside says # orbits $= \frac{1}{12}\left(3^6 + 2\cdot 3 + 2\cdot 3^2 + 4\cdot 3^3 + 3\cdot 3^4\right) = 92.$

Related questions

* What if we are only interested in rotational symmetries?
* What if we must use each color exactly 2 times?

Proofs

☆ $|\mathrm{Orb}(x)| = |\text{cosets of } \mathrm{Stab}(x)|$ (G & X could be infinite here)

Pf/ Define $f : \mathrm{Orb}(x) \to \{\text{cosets of } \mathrm{Stab}(x)\}$

$$gx \mapsto g\,\mathrm{Stab}(x)$$

Claim this is a well-defined bijection of sets. It's clearly surjective.

So note that $g\,\mathrm{Stab}(x) = h\,\mathrm{Stab}(x) \iff h^{-1}g \in \mathrm{Stab}(x)$

$\iff h^{-1}gx = x \iff gx = hx$. $\square$

☆ Orbit-stabilizer thm: If G is finite, then $\forall x \in X$, $|G| = |\mathrm{Orb}(x)||\mathrm{Stab}(x)|$

Pf/ The cosets of $\mathrm{Stab}(x)$ partition G into equally sized subsets. So

$|G| = (\text{size of each coset}) \times (\#\text{ of cosets}) = |\mathrm{Stab}(x)||\mathrm{Orb}(x)|$. $\square$ By ☆.

* Burnside: If G, S finite, $\#\text{ orbits} = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g)$.

Pf/ Consider $I := \{ (g,x) : gx = x \} \subseteq G \times X$.

Then $$|I| = \sum_{x \in X} |\underbrace{\mathrm{Stab}(x)}_{g \in G \text{ s.t. } gx = x}| = \sum_{g \in G} |\underbrace{\mathrm{Fix}(g)}_{x \in X \text{ s.t. } gx = x}|$$

Easy facts: The orbits of G partition X. If x, y are in the same orbit, then $\mathrm{Orb}(x) = \mathrm{Orb}(y)$ and $\mathrm{Stab}(x) = \mathrm{Stab}(y)$. [Check!]

Now fix $x \in X$ and note

$$\sum_{y \in \mathrm{Orb}(x)} |\mathrm{Stab}(y)| = \sum_{y \in \mathrm{Orb}(x)} |\mathrm{Stab}(x)| = |\mathrm{Orb}(x)||\mathrm{Stab}(x)| = |G|$$

Let $\mathrm{Orb}(x_1), \ldots, \mathrm{Orb}(x_t)$ be the distinct orbits of G. These partition X.

Then $$\sum_{g \in G} |\mathrm{Fix}(g)| = \sum_{x \in X} |\mathrm{Stab}(x)| = \sum_{i=1}^{t} \sum_{y \in \mathrm{Orb}(x_i)} |\mathrm{Stab}(x_i)| = t\,|G|. \quad \square$$