Math 332   Tuesday, Feb. 17

★ Quiz on Thursday.
See webpage.

* Return HW.

I. Group actions

Def. An action of a group G on a set X is a homomorphism of G into S_X, the symmetric group on X.

Equivalently, an action is a mapping

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

such that for all $g, h \in G$ and $x \in X$,

(1) $(gh)x = g(hx)$

(2) $ex = x$, where e is the identity of X.

Equivalence:

($\Rightarrow$) Given a homomorphism $\alpha : G \to S_X$, define $f : G \times X \to X$ by $f(g,x) := \alpha(g)(x)$. (Note: $\alpha(g) \in S_X$, so $\alpha(g) : X \to X$). The fact that α is a homomorphism assures (1) and (2)

($\Leftarrow$) Given $f : G \times X \to X$ as above, define $\alpha : G \to S_X$ by

$$\alpha(g) : X \to X$$
$$x \mapsto f(g,x) := gx$$

Check α is a homomorphism: For $g,h \in G$, $x \in X$,

$$\alpha(gh)(x) = (gh)x \underset{\substack{\uparrow \\ \text{property (1)}}}{=} g(hx) = g(\alpha(h)(x)) = \alpha(g)(\alpha(h)(x)) = (\alpha(g) \circ \alpha(h))(x).$$

Check $\alpha(g)$ is a bijection for all $g \in G$:

$$\alpha(g)(x) = \alpha(g)(y) \Rightarrow gx = gy \Rightarrow g^{-1}(gx) = g^{-1}(gy)$$

$$\Rightarrow (g^{-1}g)x = (g^{-1}g)y \quad \text{(property (1))}$$

$$\Rightarrow ex = ey \quad \text{(property (2))}$$

$$\Rightarrow x = y. \qquad \square$$

Note

If a group action $\alpha : G \to S_X$ is injective, then G is isomorphic to $\operatorname{im}(\alpha)$, a subgroup of the symmetric group X.

Example 20-game

We get an action of the group $G = \langle \rho, \phi \rangle = S_{20}$ on the collection of game states.

$\rho = (1,2,\ldots,20)$, $\phi = (14)(23)$

$$S_{20} \times \{\text{game states}\} \to \{\text{game states}\}$$

$\rho \cdot s$ rotates the game state $\quad s_1\, s_2 \cdots s_{20} \mapsto s_{20}\, s_1\, s_2 \cdots s_{19}$

$\phi \cdot s$ flips the game state $\quad s_1\, s_2 \cdots s_{20} \longmapsto s_4\, s_3\, s_2\, s_1\, s_5\, s_6 \cdots s_{20}$

Beginning game state: $1, 2, 3, \ldots, 20 \xrightarrow{\rho} 20, 1, 2, 3, \ldots, 19$

$20, 1, 2, 3, \ldots, 19 \xrightarrow{\phi} 3, 2, 1, 20, 4, 5, \ldots, 19$

$1, 2, 3, \ldots, 20 \longrightarrow 3, 2, 1, 20, 4, 5, \ldots, 19$

$(13)(4,5,\ldots,20) = \phi\rho$

3 in 1st position, 20 in 4th position

The game becomes: Given a game state, s, find a word w in ρ and ϕ such that $w \cdot s$ = identity state = $1, 2, 3, \ldots, 20$.

Sage: Using GAP, actually, we can find how to switch 1 and 2:

$1, 2, \ldots, 20 \longmapsto 2, 1, 3, 4, \ldots, 20$ using ρ and ϕ.

Cayley's Thm. Every group is isomorphic to a subgroup of a symmetric group.

Pf/ Let G be a group. For each $g \in G$, define the left multiplication mapping

$$\begin{aligned} l_g : G &\to G \\ h &\mapsto gh \end{aligned}.$$

It is easy to check that l_g is a group isomorphism.

Define a group action of G on itself by

$$\alpha : G \to S_G$$
$$g \mapsto l_g$$

(l_g is a bijection since its a group isomorphism).

It is easy to check α is a homomorphism. In fact, its injective: $\alpha(g) = id \Rightarrow gh = h \;\forall h \in G \Rightarrow g \cdot e = e \Rightarrow g = e$.

Thus, α is an isomorphism of G with a subgroup of S_G. ◻

Example 20-game

We can identify the collection of game states with S_{20}:

$$s_1, s_2, \ldots, s_{20} \longleftrightarrow \text{permutation } \sigma : \{1, \ldots, 20\} \to \{1, \ldots, 20\}$$
$$i \mapsto s_i$$

Then the action we described earlier is the left multiplication action from Cayley's thm.

Example $\mathbb{Z}_6 \hookrightarrow S_6$

$\ell_1(a) = 1+a$. So $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 5$, $5 \mapsto 6$

(Let 6 denote 0 in $\mathbb{Z}_6$ for ease of notation.)

Left multiplication identifies $\mathbb{Z}_6$ with $\langle (1,2,3,4,5,6) \rangle \leq S_6$:

$$1 \mapsto (1,2,3,4,5,6)$$

$$k \mapsto (1,2,3,4,5,6)^k.$$