

Math 332 Tuesday, Feb. 17

①

\* Return HW.

\* Quiz on Thursday.  
See webpage.

## I. Group actions

Def. An **action** of a group  $G$  on a set  $X$  is a homomorphism of  $G$  into  $S_X$ , the symmetric group on  $X$ .

Equivalently, an action is a mapping

$$\begin{aligned} G \times X &\rightarrow X \\ (g, x) &\mapsto gx \end{aligned}$$

such that for all  $g, h \in G$  and  $x \in X$ ,

$$(1) (gh)x = g(hx)$$

$$(2) ex = x, \text{ where } e \text{ is the identity of } X.$$

### Equivalence:

( $\Rightarrow$ ) Given a homomorphism  $\alpha: G \rightarrow S_X$ , define

$f: G \times X \rightarrow X$  by  $f(g, x) := \alpha(g)(x)$ . (Note:  $\alpha(g) \in S_X$ , so  $\alpha(g): X \rightarrow X$ ). The fact that  $\alpha$  is a homomorphism assures (1) and (2)

( $\Leftarrow$ ) Given  $f: G \times X \rightarrow X$  as above, define  $\alpha: G \rightarrow S_X$  by

$$\alpha(g): X \rightarrow X$$
$$x \mapsto f(g, x) := gx$$

Check  $\alpha$  is a homomorphism: For  $g, h \in G, x \in X$ ,

$$\alpha(gh)(x) = (gh)x \stackrel{\uparrow \text{property (1)}}{=} g(hx) = g(\alpha(h)(x)) = \alpha(g)(\alpha(h)(x)) = (\alpha(g) \circ \alpha(h))(x).$$

Check  $\alpha(g)$  is a bijection for all  $g \in G$ :

③

$$\alpha(g)(x) = \alpha(g)(y) \Rightarrow gx = gy \Rightarrow g^{-1}(gx) = g^{-1}(gy)$$

$$\Rightarrow (g^{-1}g)x = (g^{-1}g)y \quad (\text{property (1)})$$

$$\Rightarrow ex = ey \quad (\text{property (2)})$$

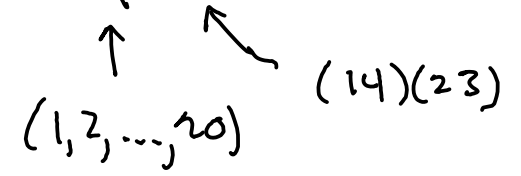
$$\Rightarrow x = y. \quad \square$$

Note

If a group action  $\alpha: G \rightarrow S_X$  is injective, then  $G$  is isomorphic to  $\text{im}(\alpha)$ , a subgroup of the symmetric group  $X$ .

# Example 20 - game

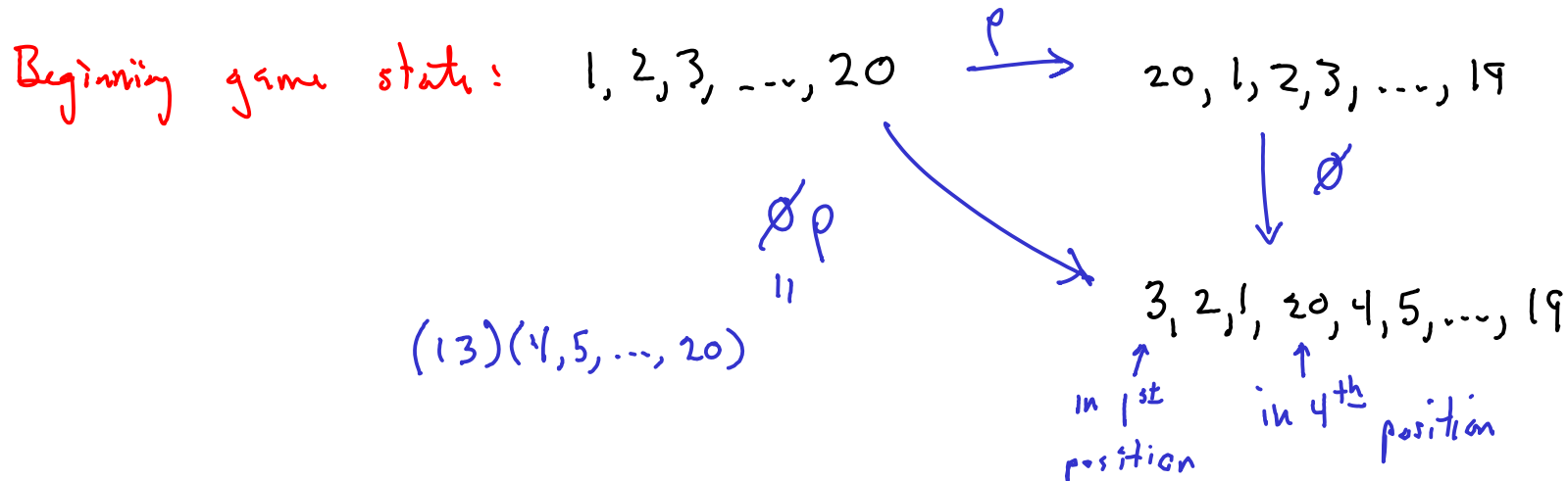
We get an action of the group  $G = \langle \rho, \phi \rangle = S_{20}$  on the collection of game states.



$$S_{20} \times \{\text{game states}\} \rightarrow \{\text{game states}\}$$

$\rho \cdot s$  rotates the game state  $s_1 s_2 \dots s_{20} \mapsto s_{20} s_1 s_2 \dots s_{19}$

$\phi \cdot s$  flips the game state  $s_1 s_2 \dots s_{20} \mapsto s_4 s_3 s_2 s_1 s_5 s_6 \dots s_{20}$



The game becomes: Given a game state,  $s$ , find a word  $w$  in  $P$  and  $\emptyset$  such that  $w \cdot s = \text{identity state} = 1, 2, 3, \dots, 20$ .

Sage: Using GAP, actually, we can find how to switch 1 and 2:  
 $1, 2, \dots, 20 \mapsto 2, 1, 3, 4, \dots, 20$  using  $P$  and  $\emptyset$ .

Cayley's Thm. Every group is isomorphic to a subgroup of a symmetric group.

Pf/ Let  $G$  be a group. For each  $g \in G$ , define the left multiplication

mapping

$$l_g: G \rightarrow G$$
$$h \mapsto gh$$

It is easy to check that  $l_g$  is a group isomorphism.

Define a group action of  $G$  on itself by

$$\alpha: G \rightarrow S_G$$

$$g \mapsto l_g$$

( $l_g$  is a bijection since its a group isomorphism).

It is easy to check  $\alpha$  is a homomorphism. In fact, its injective:  $\alpha(g) = \text{id} \Rightarrow gh = h \quad \forall h \in G \Rightarrow g \cdot e = e \Rightarrow g = e$ .

Thus,  $\alpha$  is an isomorphism of  $G$  with a subgroup of  $S_G$ .  $\square$

Example 20-game

We can identify the collection of game states with  $S_{20}$ :

$$s_1, s_2, \dots, s_{20} \leftrightarrow \text{permutation } \sigma: \{1, \dots, 20\} \rightarrow \{1, \dots, 20\}$$
$$i \mapsto s_i$$

Then the action we described earlier is the left multiplication action from Cayley's thm.

(6)

Example  $\mathbb{Z}_6 \hookrightarrow S_6$

7

$$h_1(a) = 1+a. \quad \text{So } 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 6$$

(Let  $6$  denote  $0$  in  $\mathbb{Z}_6$  for ease of notation.)

Left multiplication identifies  $\mathbb{Z}_6$  with  $\langle (1,2,3,4,5,6) \rangle \leq S_6$ :

$$1 \mapsto (1,2,3,4,5,6)$$

$$k \mapsto (1,2,3,4,5,6)^k.$$