Math 332 Thursday, Feb. 12

- ☆ Math talk
- ☆ word problem
- ☆ Tuesday HW

Quiz:

1. What is a group homomorphism?
2. Prove that if $\phi : G \to G'$ is a homomorphism of groups, then $\phi(a^{-1}) = \phi(a)^{-1}$ $\forall a \in G$.

Exercises

1. (a) Write $(1\,3\,4)$ as a product of transpositions.

$(14)(13)$

$$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 2 & 2 \\ 3 & 1 & 1 \\ 4 & 4 & 3 \end{array}$$

Another example:

$(1\,2\,3\,4\,5) = (15)(14)(13)(12)$

This example is supposed to convince you that the sign of a permutation is well-defined.

(b) Write each transposition and $(1\,3\,4)$ as permutation matrices.

$$\underbrace{\begin{bmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \end{bmatrix}}_{(134)} = \underbrace{\begin{bmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix}}_{(14)} \underbrace{\begin{bmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{bmatrix}}_{(13)}$$

$$\Rightarrow \underbrace{\det \begin{bmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \end{bmatrix}}_{1} = \underbrace{\det \begin{bmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix} \det \begin{bmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{bmatrix}}_{-1 \cdot -1}$$

2. (a) Write $(134)(752)$ as a product of transpositions.

$(14)(13)(72)(75)$

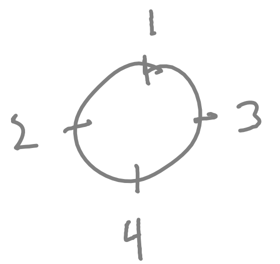
(b) Is $(1,2,\dots,k)$ odd or even? $\quad (123)(45)(6789)$?

$= (1k)(1,k-1)\cdots(12)$: even if k is odd, odd if k is even

(123) even, (45) odd, (6789) odd

Answer: even

3. order of (1342)? Why?

order $= 4$ (Think of rotating  .)

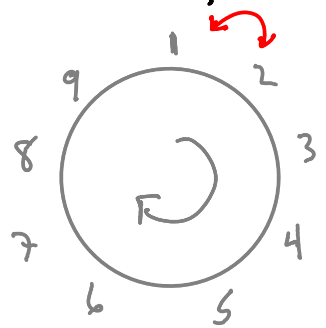
4. order of $\underbrace{(123456)}_{\text{order } 6}\underbrace{(7,8,9,10,11,12,13,14,15,16)}_{\text{order } 10}$?

5. Why do disjoint cycles commute?

Duh.

6. Why is S_n generated by (12) and $(1,2,\dots,n)$?

(Hint: can you get every transposition?)



Associated game: You can rotate the numbers or swap the 2 numbers at the position of

7. Give examples of homomorphisms of groups and their kernels.

* $\mathrm{id}: G \to G$, $a \mapsto a$ $\quad \ker(\mathrm{id}) = \{1\}$

* $\varphi: G \to H$, $a \mapsto 1$ $\quad \ker(\varphi) = G$

* $\varphi : (\mathbb{R}, +) \longrightarrow (\mathbb{R}_{>0}, \cdot)$ $\quad$ $\ker(\varphi) = 0$

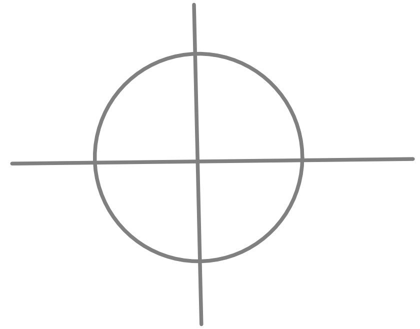
  $x \longmapsto 2^x$ $\quad$ [inverse: $x \to \lg(x) = \log_2(x)$]

* $\det : GL_n(\mathbb{R}) \longrightarrow (\mathbb{R}^*, \cdot)$ $\quad$ $\ker \varphi = SL_n(\mathbb{R})$

  $M \longmapsto \det M$

* $\mathbb{C}^* \longrightarrow (\mathbb{R}_{>}, \cdot)$ $\quad$ kernel $= \{z \in \mathbb{C} ; |z| = 1\}$

  $z \longmapsto |z|$



* $\mathrm{sgn} : S_n \longrightarrow \{\pm 1\}$ $\quad$ kernel: A_n = alternating group

  $\sigma \longmapsto \mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ even} \\ -1 & \text{if } \sigma \text{ odd} \end{cases}$

9. Why can't a group have exactly 2 elements of order 2?

Say $x, y \in G$ are distinct non-identity elements with $x^2 = y^2 = 1$.

Consider xyx. Then $(xyx)^2 = (xyx)(xyx) = 1$.

Cases: I. $xyx = x \Rightarrow x^2yx = x^2 \Rightarrow yx = 1 \Rightarrow y^2x = y \Rightarrow x = y$ $\divideontimes$

II $xyx = y \Rightarrow x^2yx = xy \Rightarrow yx = yx$.

In case II, consider xy: $(xy)^2 = xyxy = xyyx = 1$ and $xy \neq x$, $xy \neq y$.

So either $xyx \neq y$ and x, y, xyx are distinct elements of order 2

or $xy = yx$ and x, y, xy are distinct elements of order 2.