HW for Thursday: Read sections 9 + 10 (section 8 is optional).
Also look over section 45.

Quiz on this reading.

* Return quizzes and discuss.

Groups given by generators and relations

Consider the free group on S and let $R \subseteq F(S)$.
Define the group

$$G = \langle S : R \rangle$$

by starting with $F(S)$ and then defining $w = e$ for all $w \in W$.

Examples

* S a set. $F(S) = \langle S : \emptyset \rangle$

Given $\phi^2 = \rho^4 = e$

* $D_4 = \langle \phi, \rho : \phi^2, \rho^4, (\phi\rho)^2 \rangle$

Note $(\phi\rho)(\phi\rho) = e$

$\Leftrightarrow \phi\rho = \rho^{-1}\phi^{-1} = \rho^3\phi$

$= \langle \phi, \rho : \phi^2 = \rho^4 = e, \ \phi\rho = \rho^3\phi \rangle$

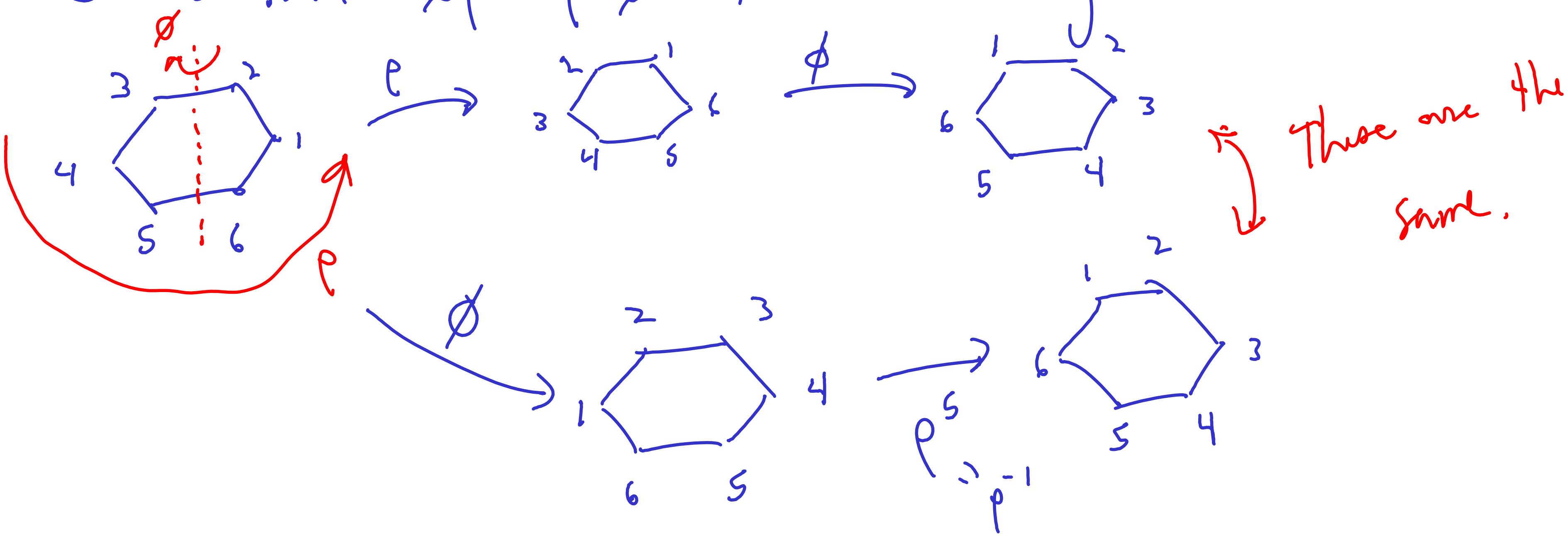
Another way of writing the relations.

* Sage code

* $D_n = \langle \phi, \rho : \phi^2 = \rho^n = e, \ \phi\rho = \rho^{n-1}\phi \rangle$.

$=$ group of symmetries of a regular n-gon

Check that $\phi\rho = \rho^5\phi$ for a hexagon



* Cyclic group of order n : $\langle a : a^n \rangle$

The word problem: $G = \langle S, R \rangle$. Given words u, v in S, are they equal in G?

* $\exists$ G as above for which there is no algorithm (computer program that can decide. ← See the wiki.

* see Web page for dictionary game.

Def. The center of a group G is $a \in G$ s.t. $ab = ba \ \forall b \in G$.

Subgroups

Def. A subgroup of a group G is a subset of G that is a group with the same operation as in G.

Tests $H \subseteq G$ is a subgroup if

1) $H \neq \emptyset$ and $ab^{-1} \in H \ \forall a, b \in H$.

or

2) $H \neq \emptyset$, $ab \in H \ \forall a, b \in H$ and $a^{-1} \in H \ \forall a \in H$.

or if G is finite

3) $e \in H$ and $ab \in H$ for all $a, b \in H$.

Reason
1) We have a binary relation on H: $H \times H \to H$
2) identity ✓
3) inverses: $a^n = e$ for some n (depending on a)
So $a^{n-1} = a^{-1}$!

Example

Prop. $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$

Pf/ Clearly, $SL_n(\mathbb{R})$ is a non-empty subset of $GL_n(\mathbb{R})$.

Let $A, B \in SL_n(\mathbb{R})$. By standard properties of the determinant, $\det(AB^{-1}) = \det(A)\det(B)^{-1} = 1$.

Hence, $AB^{-1} \in SL_n(\mathbb{R})$. $\square$

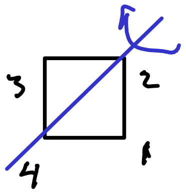
Subgroups of D_4 $= \{e, \rho, \rho^2, \rho^3, \phi, \rho\phi, \rho^2\phi, \rho^3\phi\}$

$\qquad\qquad (1234) \quad (13)(24) \quad (1432) \quad (13) \quad (14)(23) \quad (24) \quad (12)(34)$

Start with the cyclic subgroups (i.e., subgroups generated by one element):

$\langle e \rangle = \{e\}$

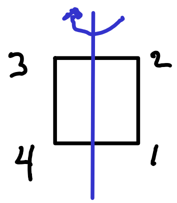
$\phi = (13)$



$\langle \phi \rangle = \{e, \phi\}$

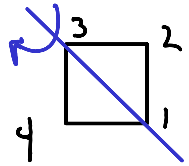
$\langle \rho \rangle = \{e, \rho, \rho^2, \rho^3\}$

$\langle \rho^2 \rangle = \{e, \rho^2\}$



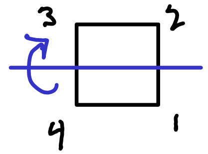
$\langle \rho\phi \rangle = \{e, \rho\phi\}$

$\rho\phi = (1234)(13) = (14)(23)$



$\langle \rho^2\phi \rangle = \{e, \rho^2\phi\}$

$\rho^2\phi = (1234)^2(13) = (13)(24)(13) = (24)$



$\langle \rho^3\phi \rangle = \{e, \rho^3\phi\}$

$\rho^3\phi = (12)(34)$

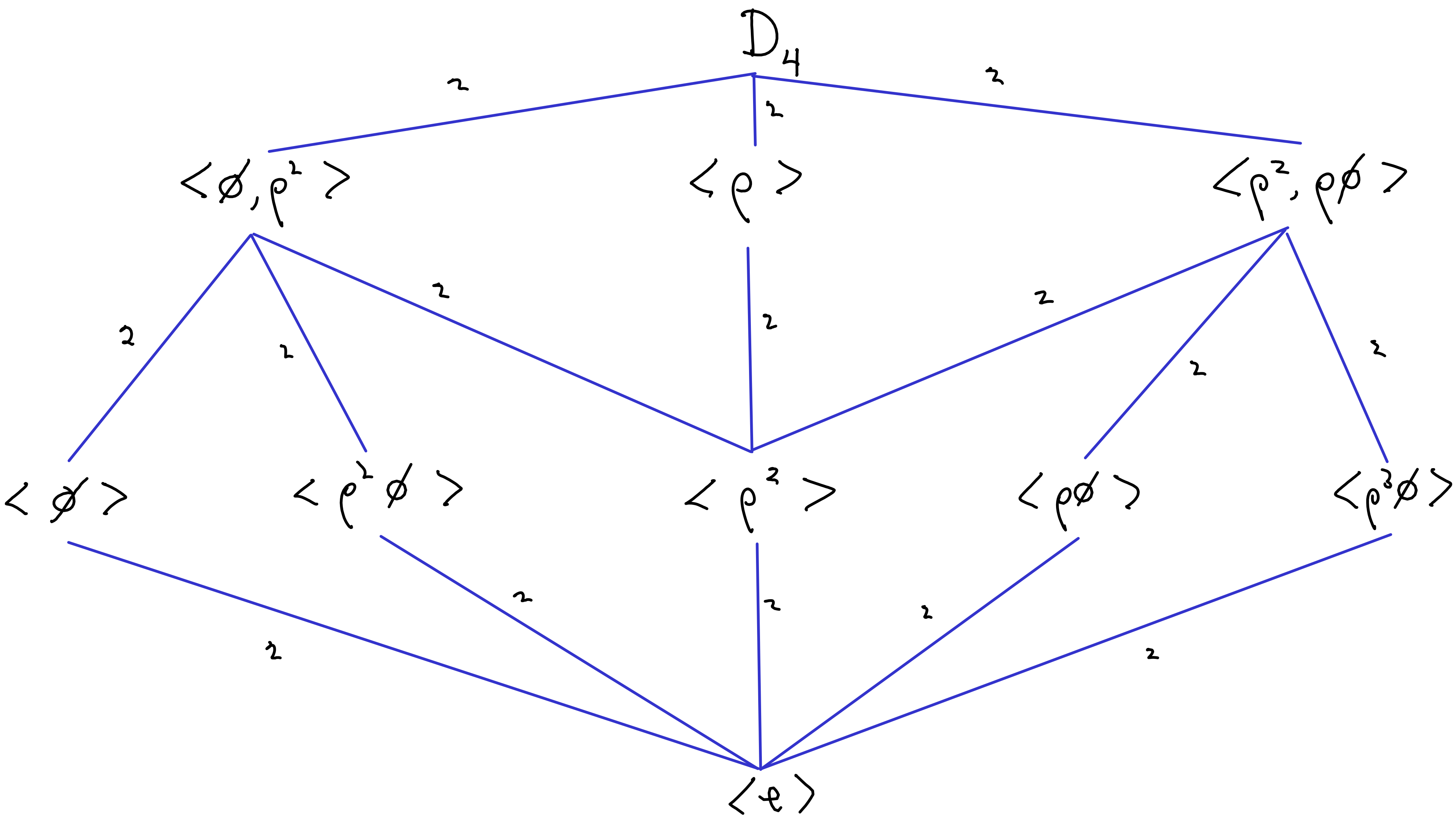
Now subgroups minimally generated by two elements

$\langle \phi, \rho^2 \rangle = \{e, \phi, \rho^2, \rho^2\phi\}$

$\langle \rho^2, \rho\phi \rangle = \{e, \rho^2, \rho\phi, \rho^3\phi\}$

$\langle \phi, \rho \rangle = D_4$

Subgroup Lattice



What are the little 2's in the lattice diagram?

Def. If G is a group and $H < G$ is a subgroup, the index of H in G

$$[G:H] = |G|/|H| \quad \text{where } |G| = \#\text{ elements of } G, \quad |H| = \#\text{ elements of } H.$$

We'll see later that this number is always an integer.

Prop If H is a subgroup of G and $a \in G$, then $aHa^{-1} := \{aha^{-1} : h \in H\}$ is a subgroup of G.

Pf/ Clearly, aHa^{-1} is nonempty. To show aHa^{-1} is closed under multiplication, let aha^{-1} and $ah'a^{-1}$ with $h, h' \in H$. Then $(aha^{-1})(ah'a^{-1}) = ahh'a^{-1} \in aHa^{-1}$ since $hh' \in H$. To show that aHa^{-1} is closed under taking inverses, take aha^{-1} with $h \in H$. Then $h^{-1} \in H \Rightarrow ah^{-1}a^{-1} \in aHa^{-1}$, and $(aha^{-1})(ah^{-1}a^{-1}) = (ah^{-1}a^{-1})(aha) = e$. $\square$

Def. Subgroups H_1 and H_2 of G are conjugate if there exists $a \in G$ such that $H_1 = a H_2 a^{-1}$.

Exercise Check that the conjugacy relation is an equivalence relation so that the above definition makes sense.

Sage

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sage: D = DihedralGroup(4)
sage: D.conjugacy_classes_subgroups()
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We can get all the subgroups by conjugating the subgroups on this list.

Exercise Identify the conjugate subgroups of D_4.