Quiz

1. What is a group?

2. Prove that the inverse of any element in a group is unique.

Putnam Problem: Suppose that a finite group G has exactly n elements of order p where p is prime. Prove that $n=0$ or p divides $n+1$.

'order' means 'number of elements'

Groups of small order

order 1 $\quad G=\{e\} \qquad e^2=e$

order 2 $\quad G=\{e,x\} \qquad x^2=x \Rightarrow x^{-1}(x^2)=x^{-1}x$

$\Rightarrow x=e \quad \#$

(Also $x^2=x \Rightarrow x^{-1}$ doesn't exist.)

So $x^2 = e$:

	e	x
e	e	x
x	x	e

$(\mathbb{Z}/2\mathbb{Z})$

order 3

$G = \{e, x, y\}$

	e	x	y
e	e	x	y
x	x		
y	y		

What is x^2?

$x^2 = x \Rightarrow x = e$ $\#$

$x^2 = e$

	e	x	y
e	e	x	y
x	x	e	y
y	y		

If $xy = e = x^2$, then $y = x$ $\#$

If $xy = x$, then $y = e$ $\#$

If $xy = y$, then $x = e$ $\#$

No good. So $x^2 = y$!

Everything else is forced now:

	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

Prop. In a multiplication table for a group, each row and each contains each group element exactly once.

Pf/ Let G be a group and $a \in G$. The row in the mult.-table corresponding to a has elements $a \cdot b$ as b varies over G. If $ab = ab'$, multiplying through by a^{-1} gives $b = b'$. Thus, no element appears more than once in any row. Now take any $c \in G$. Letting $b = a^{-1}c$, we get $a \cdot b = c$. So each element appears in each row.

The argument for columns is similar. $\square$

There's one group of order 3

	0	1	2
	e	x	x^2
e	e	x	x^2
x	x	x^2	e
x^2	x^2	e	x

$(\mathbb{Z}/3\mathbb{Z})$

order 4 $\quad G = \{e, x, y, z\}$

Two cases

1. At least one of x^2, y^2, z^2 is e.
2. None of x^2, y^2, z^2 is e

1. Say $x^2 = e$

	e	x	y	z
e	e	x	y	z
x	x	e		
y	y			
z	z			

Subcases: (i) $y^2 = e$ (ii) $y^2 \neq e$

(i) $y^2 = e$

	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

(Check associativity.)

← Klein 4-group

(ii) $y^2 \neq e$

Say $y^2 = x$

0 2 1 3 } $\mathbb{Z}/4\mathbb{Z}$

	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	x	e
z	z	y	e	x

(Check associativity.)

Say $y^2 = z$

	e	x	y	z
e	e	x	y	z
x	x	e		
y	y		z	
z	z			

z can't go here

(2) None of x^2, y^2, z^2 is e

Say $x^2 = y$

	e	x	y	z
e	e	x	y	z
x	x	y	z	e
y	y	z		x
z	z	e		y

y^2 must be e!

Done. There are two groups of order 4: $\mathbb{Z}/4\mathbb{Z}$ and a mystery group.

☆ Challenge: Find an object with this symmetry.

Free groups

Notation: For $a \in G$, $a^2 := a \cdot a$, $a^3 := a \cdot a \cdot a$, etc.

$a^{-2} := (a^{-1})^2$, $a^{-3} = (a^{-1})^3 \; \left(= (a^3)^{-1} \right)$, ...

Let S be a set. The **free group on S** is the set of symbols $s_1^{n_1} \cdots s_t^{n_t}$ where $t \in \mathbb{N}$, $n_i \in \mathbb{Z}$, $s_i \in S$, i.e., "words" in the alphabet S.

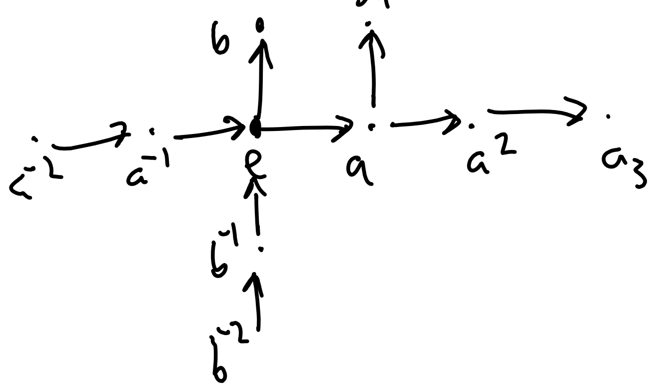
multiplication is concatenation, letting $s^m \cdot s^n = s^{m+n}$, $s^0 =$ '' = the empty word = identity

Example $G = F(\{a,b\})$

Some elements: a, b, e, $a^{-1}b^3aba^2b^{-3}a^4$

$$(a^2b^{-1})(ba^{-2}) = a^2b^{-1}ba^{-2} = a^2b^0a^{-2}$$
$$= a^2a^{-2} = a^0 = e.$$

Cayley graph?



See the Cayley graph link for today's class on our webpage.

Funny group First: Fix a dictionary.

$G = F(\{a, b, c, \dots, z\})$ / two words are equal if they're both in the dictionary and are anagrams

Example In G, cat = act. Thus, ca = ac.

apt = pat $\Rightarrow$ ap = pa.

Question What is the center of this group, i.e. which words commute with all words? Is center$(G) = G$)?