Math 332 Thursday April 9

☆ Recall correspondence: alg. sets $\leftrightarrow$ affine k-algebras

Note: varieties $\longleftrightarrow$ domains

Example A 1-1 onto mapping of algebraic sets is not necessarily an isomorphism.

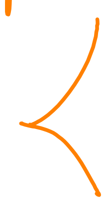
Last time:

* Algebraic sets X, Y are isomorphic if there exist polynomial maps $\phi : X \to Y$ and $\psi : Y \to X$ such that $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$.
* **Prop.** $\varphi : X \to Y$ is an isomorphism of algebraic sets iff the corresponding mapping $\varphi^* : A(Y) \to A(X)$ is an isomorphism of k-algebras.

Consider

$$\phi : \mathbb{A}^1 \longrightarrow C = Z(y^2 - x^3) \subseteq \mathbb{A}^2$$
$$t \longmapsto (t^2, t^3)$$

C is a cuspidal cubic:



ϕ is 1-1: $\phi(t) = \phi(s) \Rightarrow (t^2, t^3) = (s^2, s^3) \Rightarrow t^2 = s^2,\ t^3 = s^3$

$\Rightarrow t = s = 0$ if $t = 0$, otherwise $t = \frac{t^3}{t^2} = \frac{s^3}{s^2} = s$.

ϕ is onto: If $(p,q) \in C$, then $q^2 = p^3$. If $p = q = 0$, then $\phi(0) = (p,q)$.

Otherwise $\phi\left(\frac{q}{p}\right) = \left(\frac{q^2}{p^2}, \frac{q^3}{p^3}\right) = (p,q)$ since $q^2 = p^3$.

But the induced mapping on coordinate rings is not an isomorphism:

$$\sigma : \frac{k[x,y]}{(y^2 - x^3)} \longrightarrow k[t]$$

$$x \mapsto t^2$$

$$y \mapsto t^3$$

Note: $\operatorname{im}(\sigma) = k[t^2, t^3] \subsetneq k[t]$.

Next: For some steps towards turning more general rings into geometric objects see the end of PCMI handout 3.

PIDs & UFDs

Def. A principal ideal domain is an integral domain in which every ideal is principal. (Not "principle".)

Example (1) $\mathbb{Z}$: if $I \subseteq \mathbb{Z}$ is a nonzero ideal, let $a \in I$ be the smallest positive element of I. Claim: $I = (a)$. Let $b \in I \setminus \{0\}$. By the division algorithm,

$$b = qa + r \quad \text{with} \quad 0 \leq r < a.$$

But then $r = b - qa \in I$, so $r = 0$ by minimality of a. We've shown $I = (a)$.

Non-examples
- $k[x,y]$ is not a PID. For example, (x,y) is not principal.
- $\mathbb{Z}[x]$ is not a PID. For example, $(2,x)$ is not principal.

First goal: If k is a field, then the polynomial ring in one variable, $k[x]$, is a PID.

Def. If $f = \sum_{i=0}^{d} a_i x^i \in k[x]$ with $a_d \neq 0$, then $\deg f = d$.

We say $\deg 0 = -\infty$.

Compare with the properties of a norm: $|uv| = |u||v|$, $|u+v| \leq |u| + |v|$

Prop. (1) $\deg(fg) = \deg f + \deg g$

(2) $\deg(f+g) \leq \max\{\deg f, \deg g\}$ with equality if $\deg f \neq \deg g$.

Pf/ $f = \sum_{i=0}^{d} a_i x^i$, $g = \sum_{i=0}^{e} b_i x^i$ with $a_d \neq 0$, $b_e \neq 0$. Then

lower-order terms

(1) $fg = \sum_{k=0}^{d+e} \left(\sum_{i+j=k} a_i b_j \right) x^k = a_d b_e x^{d+e} + \text{lots}.$

(2) If $d > e$, then $f+g = a_d x^d + \text{lots}$; if $d < e$, then $f+g = a_e x^e + \text{lots}$; but if $d = e$, then $f+g = (a_d + a_e) x^{d+e} + \text{lots}$. So $\deg(f+g) \leq d+e$ with strict inequality iff $a_d + a_e = 0$.

Note: The result holds if f or g is 0, too. $\square$

Division Algorithm for $k[x]$. Let $f, g \in k[x]$ with $g \neq 0$. Then $\exists!$ $q, r \in k[x]$ such that

$$f = gq + r$$

with $\deg(r) < \deg g$.

Pf/ We first prove existence. If $\deg g > \deg f$, let $q = 0$ and $r = f$. Otherwise, we induct on $\deg f - \deg g$.

Say $f = \sum_{i=0}^{d} a_i x^i$, $g = \sum_{i=0}^{e} b_i x^i$ with $a_d \neq 0$, $b_e \neq 0$.

If $\deg f = \deg g$, let $q = a_d / b_e$ and $r = f - (a_d/b_e) g$. That takes care of the base case for induction. Now suppose $d - e > 0$. Let

$$\tilde{f} = f - {}^{a_d}/_{b_e}\, g\, x^{d-e}$$

Since $\deg \tilde{f} < \deg f$, by induction, $\exists$ $\tilde{q}, r$ such that

$$\tilde{f} = \tilde{q} g + r$$

with $\deg r < \deg g$. Then $f = \tilde{f} + a_d/b_e \, x^{d-e} g = (\tilde{q} + a_d/b_e \, x^{d-e}) g + r$.
Letting $q = \tilde{q} + a_d/b_e \, x^{d-e}$, we're done.

For uniqueness: Suppose $q_1 g + r_1 = q_2 g + r_2$ with $\deg r_i < \deg g$ for $i = 1, 2$.
Then

$$(q_1 - q_2) g = r_2 - r_1 \Rightarrow \deg(q_1 - q_2) + \deg g = \deg(r_1 - r_2) \le \max\{\deg r_1, \deg r_2\} < \deg g$$

$$\Rightarrow \deg(q_1 - q_2) < 0 \Rightarrow \deg(q_1 - q_2) = -\infty \Rightarrow q_1 = q_2 \Rightarrow r_1 = r_2. \quad \square$$