Math 332 Tuesday April 7 (See the third PCMI handout)

I. Functions.

$X \subseteq \mathbb{A}^n$: Each polynomial $f \in R = k[x_1, \dots, x_n]$ determines a mapping

$$f: X \to k$$
$$p \mapsto f(p).$$

Suppose $f, g \in R$ and $f - g \in I(X)$. If $p \in X$, then $(f-g)p = 0$, i.e. $f(p) = g(p)$. So f, g determine the same function $X \to k$.

Def. The affine coordinate ring of an algebraic set $X \subseteq \mathbb{A}^n$ is

$$A(X) := R/I(X).$$

Note: We think of elements of $A(X)$ as functions on X.

imperio! What is $(y - x^2)$?

Example $X = Z(y - x^2) \subseteq \mathbb{A}^2$. Then $A(X) = k[x,y]/(y-x^2)$

Interesting observation: $A(X) \cong k[t] = A(\mathbb{A}^1)$

Pf/ Define $\varphi: k[t] \to k[x,y]/(y-x^2)$. So $\varphi(f(t)) = f(x)$. Then
$t \mapsto x$

$f \in \ker \varphi \Rightarrow f(x) \in (y-x^2) \Rightarrow f(x) = (y-x^2)g(x,y)$ for some $g \in k[x,y]$, which can only happen if $g = f = 0$. (Otherwise, y appears in some term of $(y-x^2)g(x,y)$ but not in $f(x)$.) Thus, φ is injective. But φ is also surjective since $y = x^2$ in $k[x,y]/(y-x^2)$, given $g(x,y) \in k[x,y]$, we have $\varphi(g(t,t^2)) = g(x,y)$. $\square$

☆ Question: Which rings arise as coordinate rings of algebraic sets?

Def. Let A be a ring.

i) $a \in A$ is nilpotent if $a^m = 0$ for some m.

ii) A is reduced if it has no nonzero nilpotent elements.

iii) Let k be a field. Then A is a finitely generated k-algebra if $\exists \alpha_1, \dots, \alpha_n \in A$ such that $A = k[\alpha_1, \dots, \alpha_n] = \{ f(\alpha_1, \dots, \alpha_n) \in A : f \in k[x_1, \dots, x_n] \}$.

Example ☆ Imperio: ask class for examples.

* $\mathbb{Z}/4\mathbb{Z}$ is not reduced since $2^2 = 0$ but $2 \neq 0$ in $\mathbb{Z}/4\mathbb{Z}$.

Note A is a f.g. k-algebra iff it is the homomorphic image of a polynomial ring $k[x_1,\dots,x_n]$ for some n:

$$k[x_1,\dots,x_n] \longrightarrow A$$
$$x_i \longmapsto \alpha_i$$

Thm. Let k be algebraically closed. Then a ring A is the coordinate ring of an algebraic set X (over k) iff A is a finitely generated reduced k-algebra.

Pf/ ($\Rightarrow$) If $A = A(X)$ for some $X \subseteq \mathbb{A}^n_k$, then $A = k[x_1,\dots,x_n]/I(X)$. It is finitely generated as an algebra over k by the (equivalence classes) of $x_1,\dots,x_n$. It is reduced since $I(X)$ is radical*: $f^m = 0 \in A \Rightarrow f^m \in I(X) \overset{*}{\Rightarrow} f \in I(X) \Rightarrow f = 0 \in A$.

($\Leftarrow$) Conversely, suppose A is finitely generated as an algebra over k by $\alpha_1,\dots,\alpha_n$.

Define $\varphi : k[x_1,\dots,x_n] \to A$. Then φ is surjective. Let $J = \ker\varphi$.
$x_i \mapsto \alpha_i$

Then $A \cong k[x_1,\dots,x_n]/J$. Since A is reduced, J is radical. Hence, $J = I(Z(J))$. Letting $X = Z(J)$, we have $J = I(X)$ and $A = A(X)$. $\square$

Prop. Let $X \subseteq \mathbb{A}^n$ be an algebraic set. Then X is a variety iff $A(X)$ is a domain.

Pf/ We've seen that X is irreducible iff $I(X)$ is prime. And $I(X)$ is prime iff $R/I(X)$ is a domain (easily from the definition of "prime" and "domain"). $\square$

Def. An affine k-algebra is a finitely generated reduced k-algebra.

II. Morphism

Def. Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be algebraic sets. A morphism from X to Y is a polynomial mapping:

$$\phi : X \to Y, \quad p \mapsto (f_1(p), \dots, f_m)$$

where $f_i \in k[x_1,\dots,x_n]$ for $i = 1,\dots,m$.

Note: Not just any polynomials f_i are acceptable. The image must satisfy the equations that cut out Y.

Given a morphism of algebraic sets $\phi : X \to Y$, there is an induced mapping of coordinate rings (pulling back functions):

$$\phi^* : A(Y) \longrightarrow A(X)$$
$$f \longmapsto f \circ \phi$$

Conversely, a homomorphism of coordinate rings

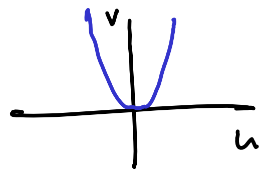
$$\sigma : k[y_1, \ldots, y_m]/I(Y) = A(Y) \longrightarrow A(X) = k[x_1, \ldots, x_n]/I(X)$$

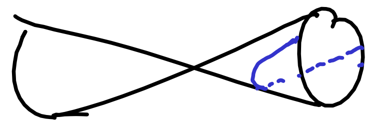
induces a morphism of algebraic sets

$$X \longrightarrow Y$$
$$p \longmapsto (f_1(p), \ldots, f_m(p))$$

where $f_i = \sigma(y_i) \in k[x_1, \ldots, x_n]$.

Example $X = Z(v-u^2) \longrightarrow Y = Z(z^2-xy)$





$(u,v) \longmapsto (1,v,u)$

Note that if $x=1$, $y=v$, $z=u$, then $z^2-xy = u^2-v=0$ for $(u,v)\in X$.

The corresponding mappings on coordinate rings:

$$\sigma: A(Y) = k[x,y,z]/(z^2-xy) \longrightarrow k[u,v]/(v-u^2)$$

$$x \longmapsto 1$$

$\left(\begin{array}{l} x: Y \to k \\ (a,b,c) \mapsto a \end{array} \right)$

$$y \longmapsto v$$

$$z \longmapsto u$$

Going back, $\sigma(x)=1$, $\sigma(y)=v$, $\sigma(z)=u$, so the induced maps on algebraic sets is:

$$X \longrightarrow Y$$

$$p \mapsto (1, v(p), u(p))$$

If $p=(p_1,p_2)$, this is $(p_1,p_2) \longmapsto (1, p_2, p_1)$, the mapping we started with.

Example The ring homomorphism

$$\sigma : \overset{\mathcal{A}(\mathbb{A}^3)\,=}{k[x,y,z]} \longrightarrow \overset{=\,\mathcal{A}(\mathbb{A}^1)}{k[t]}$$

$$x \longmapsto t^2$$
$$y \longmapsto t^3 - t$$
$$z \longmapsto t - 3$$

induces

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^3.$$
$$t \longmapsto (t^2, t^2 - 1, t - 3)$$

Example

$$\mathbb{A}^2 \longrightarrow \mathbb{A}^2$$
$$(x,y) \longmapsto (x^2 - y^3 + 2,\ x^2 - y)$$

induces

$$k[u,v] \longrightarrow k[x,y]$$
$$u \longmapsto x^2 - y^3 + 2$$
$$v \longmapsto x^2 - y$$

Prop. There is a 1-1 correspondence between morphisms of algebraic sets $X \to Y$ and ring homomorphisms $A(Y) \to A(X)$.

Def. Algebraic sets X, Y are *isomorphic* if $\exists$ morphisms

$$\phi : X \to Y \quad \text{and} \quad \psi : Y \to X$$

such that $\phi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \phi = \mathrm{id}_X$.

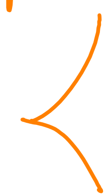
Cor. Algebraic sets X, Y are isomorphic iff their coordinate rings $A(X)$, $A(Y)$ are isomorphic. (And $\phi: X \to Y$ is iso. iff $\phi^*: A(Y) \to A(X)$ is iso.)

Thm. Let k be algebraically closed. The categories of affine k-algebras is equivalent to the category of algebraic sets over k with arrows reversed.

Warning: A morphism of algebraic sets can be 1-1 and onto without being an isomorphism:

$$\phi: \mathbb{A}^1 \longrightarrow C = Z(y^2 - x^3) \subseteq \mathbb{A}^2$$
$$t \longmapsto (t^2, t^3)$$

C is a cuspidal cubic:



ϕ is 1-1: $\phi(t) = \phi(s) \Rightarrow (t^2, t^3) = (s^2, s^3) \Rightarrow t^2 = s^2,\ t^3 = s^3$

$\Rightarrow t = s = 0$ if $t = 0$, otherwise $t = \frac{t^3}{t^2} = \frac{s^3}{s^2} = s$.

ϕ is onto: If $(p,q) \in C$, then $q^2 = p^3$. If $p = q = 0$, then $\phi(0) = (p,q)$.

Otherwise $\phi\left(\frac{q}{p}\right) = \left(\frac{q^2}{p^2}, \frac{q^3}{p^3}\right) = (p,q)$ since $q^2 = p^3$.

But the induced mapping on coordinate rings is not an isomorphism:

$$\sigma : \frac{k[x,y]}{(y^2-x^3)} \longrightarrow k[t]$$

$$x \longmapsto t^2$$

$$y \longmapsto t^3$$

Note: $\operatorname{im}(\sigma) = k[t^2,t^3] \subsetneq k[t]$.