PCMI 2008 Undergraduate Summer School

Lecture 3: Mappings

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Functions

$X \subset \mathbb{A}^n$ an algebraic set

$f \in R = k[x_1, \ldots, x_n]$ determines a function

$$\begin{aligned} f\colon X &\to k \\ p &\mapsto f(p) \end{aligned}$$

$f, g \in R$ determine the same function on X iff $f - g \in I(X)$.

Definition

The affine coordinate ring of X is

$$A(X) := R/I(X).$$

Example

$$X = Z(y - x^2)$$

$$A(X) = k[x,y]/I(X) = k[x,y]/(y - x^2)$$

$$\begin{array}{rcl} k[x,y]/(y-x^2) & \approx & k[t] \approx A(\mathbb{A}^1_k) \\ x & \mapsto & t \\ y & \mapsto & t^2 \\ x & \leftarrow & t \end{array}$$

Characterization of $A(X)$

Let A be a ring.

$a \in A$ is nilpotent if $a^m = 0$ for some m.

A is reduced if 0 is the only nilpotent element of A.

A is a finitely generated k-algebra if $k \subseteq A$ and there exists $\alpha_1, \ldots, \alpha_n \in A$ such that $A = k[\alpha_1, \ldots, \alpha_n]$.

Theorem

Suppose k is algebraically closed. A k-algebra A is the affine coordinate ring for some algebraic set *iff A is reduced and finitely generated, and it is it the affine coordinate ring of a* variety *iff A is a domain.*

Definition

An affine k-algebra is a finitely generated, reduced k-algebra.

Morphisms of algebraic sets

$X \subseteq \mathbb{A}_k^n$, $Y \subseteq \mathbb{A}_k^m$ algebraic sets.

The natural mappings (morphisms) between X and Y are polynomial mappings:

$$\begin{aligned} \phi\colon X &\to Y \\ p &\mapsto (f_1(p), \ldots, f_m(p)) \end{aligned}$$

for some $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$.

$\phi\colon X \to Y$ induces a ring homomorphism

$$\begin{array}{rcl} A(Y) & \to & A(X) \\ f & \mapsto & f \circ \phi \end{array}$$

A homomorphism

$$\sigma\colon k[y_1, \ldots, y_m]/I(Y) = A(Y) \to A(X) = k[x_1, \ldots, x_n]/I(X)$$

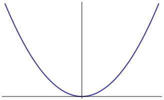
induces a morphism

$$\begin{array}{rcl} X & \to & Y \\ p & \mapsto & (f_1(p), \ldots, f_m(p)) \end{array}$$

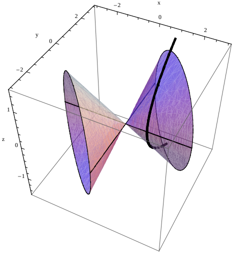
where $f_i = \sigma(y_i)$.

Example

$$\begin{array}{ccc} Z(v - u^2) & & Z(z^2 - xy) \\ \| & & \| \\ X & \longrightarrow & Y \end{array}$$



$\longrightarrow$



$$(u, v) \longmapsto (1, v, u)$$

$$\begin{array}{rcl} X & \to & Y \\ (u,v) & \mapsto & (1,v,u) \end{array}$$

$$\begin{array}{ccc} A(Y) & \to & A(X) \\ \| & & \| \\ k[x,y,z]/(z^2-xy) & \to & k[u,v] \\ x & \mapsto & 1 \\ y & \mapsto & v \\ z & \mapsto & u \end{array}$$

Proposition

There is a one-to-one correspondence between morphisms $X \to Y$ and ring homomorphisms $A(Y) \to A(X)$.

Definition

Algebraic sets X and Y are isomorphic if there are morphisms

$$\phi\colon X \to Y, \qquad \psi\colon Y \to X$$

such that $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$.

Corollary

X and Y are isomorphic iff $A(X)$ and $A(Y)$ are isomorphic as rings.

Theorem

Let k be an algebraically closed field.

The category of algebraic sets over k is equivalent to the category of affine k-algebras with arrows reversed.

Caution

A morphism of algebraic sets can be one-to-one and onto without being an isomorphism.

Example

$$\begin{array}{rcl} \mathbb{A}^1 & \to & Y = Z(y^2 - x^3) \subset \mathbb{A}^2 \\ t & \mapsto & (t^2, t^3) \end{array}$$

induces

$$\begin{array}{rcl} A(Y) = k[x, y]/(y^2 - x^3) & \to & k[t] = A(\mathbb{A}^1) \\ f(x, y) & \mapsto & f(t^2, t^3) \end{array}$$

which is not an isomorphism of rings.

Rings as geometric objects

Let k be algebraically closed.

Theorem

The maximal ideals of R are exactly the ideals of the form

$$\mathfrak{m}_p = (x_1 - a_1, \ldots, x_n - a_n)$$

for some $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$.

Corollary

Let $X \in \mathbb{A}^n$ be an algebraic set. The maximal ideals of $A(X)$ are exactly the ideals $\mathfrak{m}_p$ such that $p \in X$.

The spectrum of a ring

Let A be any ring.

Definition

The spectrum of A, denoted $\mathrm{Spec}(A)$, is the collection of prime ideals of A.

Topology

A (Zariski) closed subset of $\mathrm{Spec}(R)$ is any set of the form

$$Z(I) := \{\mathfrak{p} \text{ prime ideal of } R : \mathfrak{p} \supseteq I\}$$

A homomorphism of rings $A \to B$ induces a continuous mapping

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A).$$