Math 332 Thursday, April 2

Quiz Let $R = k[x_1, \dots, x_n]$, k a field.

1. If $S \subseteq R$, what is $Z(S)$?
2. If $X \subseteq \mathbb{A}^n$, what is $I(X)$?
3. What is a radical ideal in R?
4. What does Hilbert's Nullstellensatz say?
5. What does it mean to say R is Noetherian?

Irreducibility

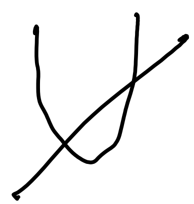
* See the handout for a discussion of irreducibility.

See Sage

Example of a reducible algebraic set

$$X = Z(x^3 - x^2y - xy + y^2)$$

$$= Z((y-x^2)(y-x)) = Z(y-x^3) \cup Z(y-x)$$



X

Prop. An algebraic set X is irreducible iff $I(X)$ is prime.

Pf/ First suppose X is reducible. Then $X = X_1 \cup X_2$ with $X_i \subsetneq X$.
Now, $X_i \subsetneq X \Rightarrow I(X_i) \supseteq I(X)$. If $I(X_i) = I(X)$, then
$X_i = Z(I(X_i)) = Z(I(X)) = X$ since X_i and X are algebraic sets.
So $I(X_i) \supsetneq I(X)$. Take $f_i \in I(X_i) \setminus I(X)$ for $i=1,2$.
Then $f_1 f_2 \in I(X)$. Check: Let $p \in X = X_1 \cup X_2$. We may
assume $p \in X_1$. Then $(f_1 f_2)(p) = f_1(p) f_2(p) = 0 \cdot f_2(p) = 0$. ✓
So $f_1 f_2 \in I(X)$ but $f_1 \notin I(X)$ and $f_2 \notin I(X)$. So $I(X)$ is prime.

Conversely, suppose $I(X)$ is not prime. Take $f_1 \cdot f_2 \in I(X)$ s.t. $f_i \notin I(X)$
for $i=1,2$. Let $V_i = Z(f_i)$ and note $(f_i) \subseteq I(X) \Rightarrow V_i = Z(f_i) \supseteq Z(I(X))$
$= X$. Let $X_i = V_i \cap X$. Then $V_1 \cup V_2 \supseteq X \Rightarrow X \cap (V_1 \cap V_2) = X$

$\Rightarrow X_1 \cup X_2 = X$. If $X_i = X$, then $f_i \in I(X_i) = I(X)$, a contradiction. So $X_i \subsetneq X$ for $i = 1, 2$. $\square$

Lemma Let B be a Noetherian ring, and let $\mathcal{A}$ be a non-empty collection of ideals of B. Then $\mathcal{A}$ has a maximal element with respect to inclusion.

Pf/ Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be a chain of ideals from $\mathcal{A}$. Define $I = \bigcup_{i \geq 1} I_i$. Then I is an ideal of B. Since B is Noetherian, I is finitely generated, i.e. $\exists\, b_1, \ldots, b_s \in B$ such that

$$I = (b_1, \ldots, b_s).$$

Since $I = \bigcup I_i$ and the I_i's are nested, $\exists\, N$ s.t. $b_1, \ldots, b_s \in I_N$. But then our chain looks like $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_N = I_{N+1} = \cdots$.

If $\mathcal{A}$ had no maximal element with respect to inclusion, we could construct a strictly increasing, infinite chain of ideals of A. However, we've just shown that can't happen. $\square$

Cor. Every nonempty collection of algebraic sets in $\mathbb{A}^n$ has a minimal element with respect to inclusion.

Pf/ Let $\{X_i\}$ be a collection of algebraic sets. Then $\{I(X_i)\}$ is a collection of ideals of $R = k[x_1, \dots, x_n]$, a Noetherian ring by the Hilbert basis theorem. Thus, $\{I(X_i)\}$ has a maximal element, $I(X_N)$. Claim: X_N is a minimal element of $\{X_i\}$. To see this, suppose $X_i \subseteq X_N$ for some i. Then $I(X_i) \supseteq I(X_N)$. By maximality of $I(X_N)$, we have $I(X_i) = I(X_N)$. Whence, $X_i = Z(I(X_i)) = Z(I(X_N)) = X_N$. $\square$

Proof of decomposition theorem/ (See the 2nd PCMI handout for a statement of the decomposition theorem.)

Let $\mathcal{B} = \{$ algebraic sets of $\mathbb{A}^n$ that are not unions of a finite number of irreds$\}$. If $\mathcal{B} = \emptyset$, we are done. If not, we've just seen that $\mathcal{B}$ has a minimal element, say X. Is X irreducible? No, since then $X \in \mathcal{B}$. So we can write $X = X_1 \cup X_2$ with $X_i \subsetneq X$. By minimality, $X_1, X_2 \notin \mathcal{B}$. Thus, they are unions of a finite number of irreducibles. But then so is X. Thus, we must have $\mathcal{B} = \emptyset$. $\square$

Def. An **algebraic variety** is an irreducible algebraic set.