Math 332 Tuesday March 31

* Thursday quiz on alg. geo. from last Thursday and today.

* HW for next week: A preliminary version is posted. It may be revised by the end of the week. They are all good problems, though!

Return HW and review

Algebraic Geometry

$R = k[x_1, \dots, x_n]$, $I \subseteq R$ ideal

$$\mathcal{Z}(I) = \{ p \in \mathbb{A}^n_k : f(p) = 0 \ \forall f \in I \}$$

$$X \subseteq \mathbb{A}^n_k, \quad I(X) = \{ f \in R : f(X) = 0 \}$$

Alg.		Geo.
J	$\longmapsto$	$\mathcal{Z}(J)$
$I(X)$	$\longleftarrow$	X

Prop. Let J, K be ideals in R, and let X, Y be subsets of $\mathbb{A}^n_k$. Then

(1) $J \subseteq K \Rightarrow \mathcal{Z}(J) \supseteq \mathcal{Z}(K)$.

(2) $X \subseteq Y \Rightarrow I(X) \supseteq \mathcal{Z}(Y)$

(3) $I(\mathcal{Z}(J)) \supseteq J$ $\leftarrow$ equality not necessary. Example: $J = (x^2) \subseteq k[x]$. Then $I(\mathcal{Z}(J)) = (x)$.

(4) $\mathcal{Z}(I(X)) \supseteq X$ $\leftarrow$ equality not necessary. Example: $X = \mathbb{Z} \subseteq \mathbb{R}$. Then $\mathcal{Z}(I(X)) = \mathbb{R}$.

(5) $Z(I(Z(J))) = Z(J)$

(6) $I(Z(I(X))) = I(X)$

Proof of (5)/ $\supseteq$ Let $p \in Z(J)$, and let $f \in I(Z(J))$. Then $f(p) = 0$, so $p \in Z(I(Z(J)))$.

$\subseteq$ Conversely, let $p \in Z(I(Z(J)))$, and let $f \in J$. Claim: $f \in I(Z(J))$. To see this, let $q \in Z(J)$. Then $f(q) = 0$. Since $f \in I(Z(J))$ and $p \in Z(J)$, we get $f(p) = 0$. $\square$

Def. $X \subseteq \mathbb{A}^n$ is an *algebraic set* if $X = I(J)$ for some $J < R$.

Cor. If X is an algebraic set, then $Z(I(X)) = X$.

Pf/ $X = Z(J)$ for some J. See part (5), above. $\square$

Thm. (Hilbert's Nullstellensatz) If k is algebraically closed, then for any ideal $J \subset R = k[x_1, \ldots, x_n]$, $I(Z(J)) = \mathrm{rad}(J) = \{ f \in R : f^m \in J$ for some $m \in \mathbb{Z}_{>0} \}$.

Cor. Over an algebraically closed field, $\exists$ bijection between algebraic sets and radical ideals.

Hilbert basis thm.

If $f, g \in R$, then $Z(f,g) = Z(f) \cap Z(g)$.

Def. A *hypersurface* is any algebraic set of the form $Z(f)$ with $f \in R$, non constant (i.e. $f \notin k$).

Question Is every algebraic set the intersection of finitely many hypersurfaces?

Translation into algebra: Is every ideal $J \subset R$ finitely generated?

$\exists f_1, \ldots, f_s \in R$ s.t. $J = (f_1, \ldots, f_s)$?

Example What about $(y-x, y-x^2, y-x^3, \ldots) \subseteq k[x,y]$?

* Now see the handout for the set-up and proof of the Hilbert Basis theorem.