Math 332 Tuesday 3/24/09

Today: * review homework
* overview of rings

Note: Sage 3.4 is out.

Quiz

1. What is a ring?

2. What is an ideal?

3. What is the characteristic of a ring?

Thursday's quiz: Sections 27 – 34.

Go over HW: (especially the last problem)

Rings (Summary of reading)

Def. A ring is a set R with two binary operations $+, \cdot$ such that

(i) $(R, +)$ is an abelian group with identity 0.

(ii) $\cdot$ is associative and distributive over $+$ on the left and right.

Examples

1. Summarize Irena's examples.

2. Interesting noncommutative rings

indeterminates

* Let G be any group. $\mathbb{C}[G] = \mathbb{C}[x_g : g \in G] = \{ \sum' a_g x_g : g \in G, a_g \in \mathbb{C} \}$

Multiplication as for polynomials, but $x_g \cdot x_h = x_{gh}$. So, in general,

$$x_g x_h \neq x_h x_g$$

This is called the group ring for G.

* $M_n(R)$ = $n \times n$ matrices over a ring R

* $k[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$ = polynomials in $x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ where $\frac{\partial}{\partial x_i}$ acts as a differential operator.

Example:
$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$
$$= x\frac{\partial}{\partial x}x\frac{\partial}{\partial x} + x\frac{\partial}{\partial x}y\frac{\partial}{\partial y} + y\frac{\partial}{\partial y}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}y\frac{\partial}{\partial y}$$
$$= x\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial y^2}.$$

Big problem: Classify all rings.

First definitions

A ring with identity is a ring with multiplicative identity (usually denoted 1). We always take $1 \neq 0$.

A ring is commutative if it's multiplication is commutative.

A subring of a ring R is a subset $S \subseteq R$ that is a ring (with the operations of R).

Subring test: $S \subseteq R$ is a subring if S is nonempty and closed under multiplication and subtraction.

A homomorphism between rings S, T is a function $\phi : S \to T$ such that $f(s+s') = f(s) + f(s')$ and $f(ss') = f(s)f(s')$. [If both S and T have identities, we usually require $f(1_S) = 1_T$.]

A left (resp. right) ideal in a ring R is an additive subgroup $I \subseteq R$ such that $ri \in I$ $\forall r \in R, \forall i \in I$, i.e. $RI \subseteq I$ (resp. $ir \in I$ $\forall r \in R, \forall i \in I$, i.e. $IR \subseteq I$). An ideal is a subset that is both a left and right ideal.

The **kernel** of a ring homomorphism $\varphi : S \to T$ is

$$\ker \varphi = \{ s \in S : \varphi(s) = 0 \}.$$

Prop. The kernel of a ring homomorphism is an ideal.

Pf/ Let $\varphi : S \to T$ be a homomorphism of rings. Since φ is a homomorphism of additive groups, $\ker \varphi$ is a subgroup. Let $s \in S$ and $x \in \ker \varphi$. Then $\varphi(sx) = \varphi(s)\varphi(x) = \varphi(s) \cdot 0 = 0$, and $\varphi(xs) = \varphi(x)\varphi(s) = 0 \cdot \varphi(s) = 0$. $\square$

std. property of rings.

Let $I \leq R$ be an ideal. The **quotient ring** or **factor ring** corresponding to I is the quotient as additive groups

$$R/I = \{ a + I : a \in R \}$$

with ring structure $(a+I)(b+I) = ab + I \quad \forall a, b \in R$.

Check that multiplication is well-defined:

Suppose $a+I = a'+I$. We would like to see that $ab+I = a'b+I$ $\forall b \in R$. Well, $a+I = a'+I \Rightarrow a-a' \in I \Rightarrow (a-a')b \in I$ (since I is an right ideal) $\Rightarrow ab - a'b \in I \Rightarrow ab+I = a'b+I$. Similarly, if $b+I = b'+I$, then $ab+I = ab'+I$. (So we have used that I is a two-sided ideal). $\square$

Prop. $I \subseteq R$ is an ideal iff $I = \ker \varphi$ for a ring homomorphism φ.

Pf/ We've already seen "$\Leftarrow$". For "$\Rightarrow$", consider the quotient map

$$\begin{aligned} R &\to R/I \\ a &\mapsto a+I. \end{aligned}$$ $\square$

First isomorphism theorem. If $\varphi: S \to T$ is a ring homomorphism, then $S/\ker\varphi \cong \operatorname{im}\varphi$. $\leftarrow$ ($\operatorname{im}\varphi$ is a subring)

Pf/ Duh. $\square$

Closely related idea. The ring homomorphism $S \to T$ factors into a surjection followed by an injection:

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & T \\ & S/\ker\varphi \arrow[ur, hook, "\iota"'] & \end{tikzcd} \qquad \varphi = \iota \circ \pi .$$

A sequence of ring homomorphisms $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ is exact at S if $\operatorname{im}\varphi = \ker\psi$. Short exact sequence (ses) of rings:

$$0 \to R' \xrightarrow{\varphi} R \xrightarrow{\psi} R'' \to 0 .$$

In this case, $R' \approx \operatorname{im}\varphi = \ker\psi$ and $R'' \approx R/\ker\psi$.