Math 332 Thursday March 12

☆ Thursday talk

Semidirect products

Suppose the short exact sequence of groups

$$1 \to N \xrightarrow{\phi} G \xrightarrow{\psi} H \to 1$$

"mapping of groups" = "homomorphism"

has a splitting, i.e. a mapping $j: H \to G$ such that $\psi \circ j = id_H$.

Then j must be injective: $j(h) = j(h') \Rightarrow \psi j(h) = \psi j(h') \Rightarrow h = h'$.

Since ϕ is injective, it is an isomorphism onto its image, which is $\ker \psi$, hence normal in G. Letting H act on N by conjugation gives an automorphism of N

$$\alpha : H \longrightarrow Aut(N) \quad \text{where} \quad c_h : N \longrightarrow N$$

$$h \longmapsto c_h \qquad\qquad n \longmapsto j(h)\phi(n)j(h)^{-1}$$

One may check that α is a homomorphism of groups.

Example $G = N \times H$

$$1 \to N \xrightarrow{\phi} N \times H \xrightarrow{\psi} H \to 1 \qquad N \times H \xleftarrow{j} H$$

$$n \mapsto (n,1) \qquad (1,h) \leftarrow h$$

$$(n,h) \mapsto h$$

In this case $\alpha : H \to \mathrm{Aut}(N)$ where $c_h(n) = (1,h)(n,1)(h,1)^{-1}$

$h \mapsto c_h$ $\qquad = (n,1)$.

Thus, $\alpha(h) = id_N \ \forall h \in H$.

Example $G = D_4 = \langle \phi, \rho : \phi^2 = \rho^4 = 1,\ \rho\phi = \phi\rho^3 \rangle$, $H = \langle \phi \rangle$, $N = \langle \rho \rangle$.

$$1 \to N \to G \to H \to 1$$

$$\rho \mapsto 1$$

$$\phi \mapsto \phi$$

(Check this $G \to H$ is a well-defined homomorphism. For instance, $\rho\phi = \phi\rho^3$ and $\rho\phi \mapsto \phi$, $\phi\rho^3 \mapsto \phi$.)

The mapping $j : H \to G$, $\phi \mapsto \phi$ gives a splitting.

The corresponding mapping $H \to \mathrm{Aut}(N)$ is given by $h \mapsto c_h$ where $c_h(n) = hnh^{-1}$. The image of H here consists of 2 automorphisms: $c_1 = \mathrm{id}_N$ and c_ϕ = conjugation by ϕ, and $c_\phi(\rho^i) = \phi\rho^i\phi^{-1} = \phi\rho^i\phi = \rho^{3i}$. Note that since $N \cong \mathbb{Z}_4$, a cyclic group with 2 generators, 1 and 3, we get $|\mathrm{Aut}(N)| = 2$. So $H \to \mathrm{Aut}(N)$ in this case is an isomorphism.

Now turn things around

Def. Let N and H be groups, and let $\alpha : H \to \mathrm{Aut}(N)$ be a homomorphism. The semidirect product of N and H is the Cartesian product $N \times H$ with multiplication given by
$$(n_1, h_1)\cdot(n_2, h_2) = (n_1\, \alpha(h_1)(n_2),\ h_1 h_2).$$

Notation: $N \rtimes H$ or $N \rtimes_\alpha H$, if α is not clear from context.

Prop. With notation as above,

(1) $N \rtimes H$ is a group.

(2) $\exists$ a split exact sequence $1 \to N \to N \rtimes H \rightleftarrows H \to 1$

(and thus N is a normal subgroup of $N \rtimes H$).

Pf/ HW. $\square$

Prop. Given any split exact sequence $1 \to N \to G \rightleftarrows H \to 1$,

we have seen there is a natural mapping $\alpha : H \to \mathrm{Aut}(N)$ given,

roughly, by conjugation by elements of H. Then $G \approx N \rtimes_\alpha H$.

Pf/ HW. $\square$

Example There are 2 homomorphisms $\alpha : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_4) = \{ \underset{\gamma_1}{id_{\mathbb{Z}_4}}, \underset{\gamma_3(1)=3}{\gamma_3} \} \approx \mathbb{Z}_2$

(1) $\alpha(0) = \alpha(1) = id_{\mathbb{Z}_4} : 1 \mapsto 1$. In this case $\mathbb{Z}_2 \rtimes_\alpha \mathbb{Z}_4 = \mathbb{Z}_2 \times \mathbb{Z}_4$.

(2) $\alpha(0) = id_{\mathbb{Z}_4}$, $\alpha(1) = \gamma_3$. In this case, $\mathbb{Z}_2 \rtimes_\alpha \mathbb{Z}_4 \approx D_4$

Wreath Products

Def. Let $P < S_n$, and let G be a group. The *wreath product* of G with P is

$$G \wr P_n := G^{\times n} \rtimes P$$

where $P \to \mathrm{Aut}(G^{\times n})$ by permuting the components of $G^{\times n}$, i.e. for $\sigma \in P$, $\sigma(g_1, \ldots, g_n) = (g_{\sigma(1)}, \ldots, g_{\sigma(n)})$. Thus,

$$((g_1, \ldots, g_n), \sigma)((g_1', \ldots, g_n'), \tau) = ((g_1 g_{\sigma(1)}', \ldots, g_n g_{\sigma(n)}'), \sigma\tau).$$

One way of thinking about $G \wr P$ is to identify P with a group of $n \times n$ permutation matrices. If $M \in P$, then wherever there is an entry of 1 in M, replace it by some element of G. This gives an $n \times n$ matrix with whose nonzero entries that are elements of G. Multiply two such matrices in the natural way. (See the example, below.)

Example Let $G = \mathbb{Z}$ and let $P = \langle (123) \rangle < S_3$.

We think of P as the set of permutation matrices $\left\{ \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix} \right\}$.

Elements of $\mathbb{Z} \wr P$ have the form

$$\begin{bmatrix} a&0&0\\0&b&0\\0&0&c \end{bmatrix}, \begin{bmatrix} 0&0&a'\\b'&0&0\\0&c'&0 \end{bmatrix}, \begin{bmatrix} 0&a''&0\\0&0&b''\\c''&0&0 \end{bmatrix} \quad \text{where } a,b,c,a',b',c',a'',b'',c'' \in \mathbb{Z}.$$

For example,

$$\begin{bmatrix} 2&0&0\\0&1&0\\0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&5\\3&0&0\\0&9&0 \end{bmatrix} = \begin{bmatrix} 0&0&7\\4&0&0\\0&9&0 \end{bmatrix}$$

note that $0+9=9$

Def. Let N, H be groups. Then $N \wr H := N \wr P(H)$ where $P(H)$ is the group H, thought of as a permutation group via Cayley (H acts on itself via left multiplication, and left multiplication permutes the elements of H).

The Jordan-Hölder Theorem

Let G be a finite group. A composition series for G is a maximal chain of subgroups

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_n = G$$

"Maximal" means that one could not refine the chain by adding more subgroups, i.e. for each i, if $\exists\, \tilde{H}$ s.t. $H_i \triangleleft \tilde{H} \triangleleft H_{i+1}$, then $\tilde{H} = H_i$ or $\tilde{H} = H_{i+1}$.

The factors of a composition series are the groups H_{i+1}/H_i. (Maximality implies the factors are simple groups.) The length of the composition series is n.

Thm. (Jordan-Hölder) The length and the list of composition factors (up to a permutation of the list) is independent of G.

Example Let C_n be the cyclic group of order n

Composition Series	Composition factors	length
$C_1 \triangleleft C_2 \triangleleft C_4 \triangleleft C_{12}$	C_2, C_2, C_3	3
$C_1 \triangleleft C_2 \triangleleft C_6 \triangleleft C_{12}$	C_2, C_3, C_2	3
$C_1 \triangleleft C_3 \triangleleft C_6 \triangleleft C_{12}$	C_3, C_2, C_2	3