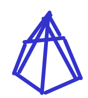
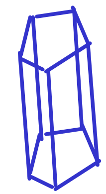
Math 332 Tuesday March 2

Today:
* review HW from last week
* Normal subgroups
* factor groups

1

Thursday Quiz The quiz covers the material in sections 21 – 25, inclusive. Know the statements of results. Be able to prove the first isomorphism theorem (Thm 24.1).

Fun Fact The finite groups of rotations of $\mathbb{R}^3$ [$SO(3) = \{M \in GL(\mathbb{R}^3) : MM^t = I_n \text{ and } \det M = 1\}$]

two infinite families:
- C_n for $n = 1, 2, \ldots$: cyclic group of rotations of a regular pyramid 
- D_n for $n = 1, 2, \ldots$: dihedral group of rotations of a regular cylinder 

three finite exceptional groups:
- T : rotational symmetries of a tetrahedron
- O : rotational symmetries of an octahedron (or cube)
- I : rotational symmetries of an icosahedron (or dodecahedron)

I. Normal Subgroups

Let G be a group and let $\mathcal{S}(G)$ be the set of subgroups of G. Then G acts on $\mathcal{S}(G)$ by conjugation:

$$G \times \mathcal{S}(G) \to \mathcal{S}(G)$$
$$(g, H) \mapsto g^{-1}Hg$$

The Sage command G.conjugacy_classes_subgroups() returns an element from each orbit of this action. All the subgroups in a given orbit are conjugate, hence, isomorphic.

Def. $H < G$ is a normal subgroup of G, denoted by $H \lhd G$, if $gH = Hg$ for all $g \in G$.

Equivalently: $gHg^{-1} = H \;\; \forall g \in G$, i.e. the orbit of H under the above action consists of a single element.

Examples

1) If G is abelian, then each of its subgroups is normal.
2) $Z(G) \triangleleft G$ (the center is normal).
3) A_n is normal in S_n. Reason: $A_n = \{\text{even permutations}\}$. If $g \in S_n$ is even, then $gA_n = A_n g = A_n$. If $g \in S_n$ is odd, then $gA_n = A_n g = \{\text{odd permutations}\}$.
4) More generally, $[G:H] = 2 \Rightarrow H \triangleleft G$. Pf/ HW. $\square$
5) S_4 has 4 normal subgroups. Sage code:

```
gap(SymmetricGroup(4)).NormalSubgroups()
```

6) For $n \geq 5$, S_n has 3 normal subgroups: $\{()\}$, A_n, S_n, and A_n has 2: $\{()\}$, A_n.
7) S_3 has a non-normal subgroup. (See this week's HW.)

Def. A group is simple if it has no nontrivial proper subgroups.
(See wiki page on the finite simple groups.)

Prop. A_n is simple for $n \geq 5$. **Pf/** Exercise for the reader. See Irena's notes for hints. $\square$

II. Factor Groups

Def. Let $H \lhd G$. Define the quotient group, denoted G/H, by
$$G/H = \{aH : a \in G\}$$ with multiplication $(aH)(bH) = (ab)H$.

Claim: This multiplication is well-defined. **Check /** Say $aH = a'H$ and $bH = b'H$.
So $a^{-1}a' \in H$ and $b^{-1}b' \in H$. Now $(ab)H = (a'b')H$ iff $(ab)^{-1}(a'b') \in H$.

But $(ab)^{-1}(a'b') = b^{-1}\underbrace{(a^{-1}a')}_{=:h \in H}b' = b^{-1}hb' = b^{-1}b'h'$ for some $h' \in H$

Since $b'H = Hb'$. Finally, $b^{-1}b' \in H$, $h' \in H \Rightarrow b^{-1}b'h' \in H$. $\square$

Examples (1) For $n \in \mathbb{Z}$, $\quad n\mathbb{Z} = \{na : a \in \mathbb{Z}\} \lhd \mathbb{Z}$.

$$\mathbb{Z}/n\mathbb{Z} = \{n\mathbb{Z}, 1+n\mathbb{Z}, 2+n\mathbb{Z}, \dots, (n-1)+n\mathbb{Z}\}$$
$$=: \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}\}.$$

(2) $D_4 = \langle \rho, \phi : \rho^4 = \phi^2 = 1, \ \phi\rho = \rho^3\phi \}$, $\quad N = \langle \rho \rangle \lhd D_4$

Check: $\rho^k N = N\rho^k = N \quad \forall k.$

$\rho^6 = \rho^2 \qquad \rho^9 = \rho$

$$\phi N = \{\phi, \phi\rho, \phi\rho^2, \phi\rho^3\} = \{\phi, \rho^3\phi, \rho^2\phi, \rho\phi\} = N\phi$$

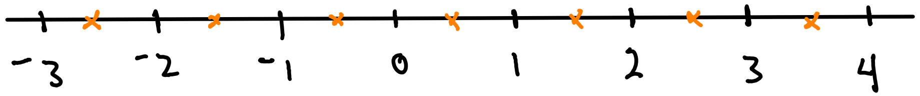
$$\rho^k\phi N = \rho^k N\phi = N\rho^k\phi.$$

$$D_4/N = \{N, \phi N\}$$

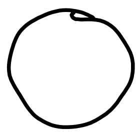
*	N	ϕN
N	N	ϕN
ϕN	ϕN	N

Recall the structure of the multiplication table for D_4, earlier.

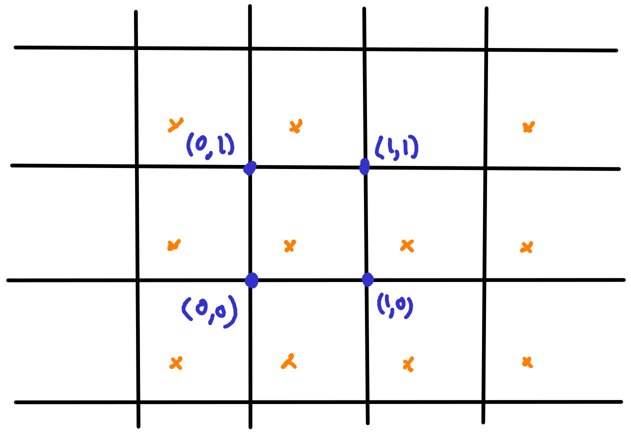
(3) $\mathbb{R}/\mathbb{Z}$



The points marked with × give the coset $\frac{1}{2} + \mathbb{Z}$

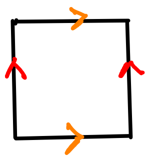
Modding out by $\mathbb{Z}$, identifies these points. So a good geometric picture of $\mathbb{R}/\mathbb{Z}$ is a circle  .

(4) $\mathbb{R}^2/\mathbb{Z}^2$



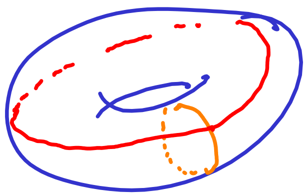
The points marked × show the coset $(\frac{1}{4},\frac{1}{4}) + \mathbb{Z}^2$. Modding out by $\mathbb{Z}^2$ identifies these points.

Points along the edges are equivalent mod $\mathbb{Z}^2$.



Every point is equivalent, mod $\mathbb{Z}^2$ to a point in any fixed square.

Identify the equivalent points along the boundary of the square to get a set containing a unique representative of each equivalence class:

 $\approx \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$