Math 332 S.E.S. HW

short exact sequence: $1 \to N \xrightarrow{\varphi} G \xrightarrow{\psi} H \to 1$

1. ψ is surjective. Pf/ Being exact at H means $\operatorname{Im}\psi = \ker(H \to 1) = H$. $\square$

2. φ is injective Pf/ Being exact at N means $\ker \varphi = \operatorname{im}(1 \to N) = 1$, and we've seen that $\ker \varphi = \{1\} \Rightarrow \varphi$ injective. $\square$

3. $N \trianglelefteq G$. Pf/ Here, we've replaced N by $\varphi(N)$ and φ by the inclusion mapping to assume $N < G$. Let $g \in G$ and $n \in N$. Since $\ker \psi = N$, we have $\psi(gng^{-1}) = \psi(g)\psi(n)\psi(g^{-1}) = \psi(g)\psi(g^{-1}) = \psi(g)\psi(g)^{-1} = 1 \Rightarrow$ $gng^{-1} \in N$. $\square$

4. $H \cong G/N$. Pf/ This is the first isomorphism theorem.

5. If G is abelian and there exists a homomorphism $j: H \to G$ s.t. $\psi \circ j = \mathrm{id}_H$, then $G \cong N \times H$.

Pf/ Define $f : N \times H \to G$.
$$(n, h) \mapsto n \cdot j(h)$$

* f is a homomorphism.

$$f((n,h)(n',h')) = f(nn', hh') = nn' j(hh') = [n j(h)][n' j(h')] = f(n,h)\, f(n',h'),$$

using the fact that j is a homomorphism and G is abelian. $\square$

* f is injective.

$1 = f(n,h) = n j(h) \Rightarrow j(h) = n^{-1} \in N = \ker \Psi \Rightarrow h = (\Psi \circ j)(h) = 1$, and then $1 = n j(h) \Rightarrow 1 = n j(1) = n$. So $(n,h) = (1,1)$, the identity of $N \times H$. $\square$

* f is surjective.

Given $g \in G$, note that $g \cdot j(\Psi(g))^{-1} \in \ker \Psi$ since $\Psi(g \cdot (j\Psi(g))^{-1}) = \Psi(g) [(\Psi j)(\Psi(g))]^{-1} = \Psi(g)\, \Psi(g)^{-1} = 1$. Thus, $(g \cdot j(\Psi(g))^{-1}, \Psi(g)) \in N \times H$ and $f(g \cdot j(\Psi(g))^{-1}, \Psi(g)) = g\, j(\Psi(g))^{-1} j(\Psi(g)) = g$. $\square$