Math 332 HW 9 Solutions

1. a) $S \subseteq R$, $I = I(S) \Rightarrow Z(S) = Z(I)$.

Pf/ Let $p \in Z(S)$ and let $f \in I$. Then $f = \sum' g_i s_i$ with $s_i \in S$.
So $f(p) = \sum' g_i(p) s_i(p) = \sum' g_i(p) \cdot 0 = 0$. Hence, $Z(S) \subseteq Z(I)$.
Conversely, let $p \in Z(I)$ and $s \in S$. Then $S \subseteq I \Rightarrow s \in I \Rightarrow s(p) = 0$. $\square$

b) $S \subseteq T \subseteq R \Rightarrow Z(S) \supseteq Z(T)$.

Pf/ Let $p \in Z(T)$ and $s \in S$. Then $S \subseteq T \Rightarrow s \in T \Rightarrow s(p) = 0$. $\square$

c) $X \subseteq Y \subseteq \mathbb{A}^n \Rightarrow I(X) \supseteq I(Y)$.

Pf/ Let $f \in I(Y)$ and $p \in X$. Then $X \subseteq Y \Rightarrow p \in Y \Rightarrow f(p) = 0$. $\square$

d) $S \subseteq R$, $X \subseteq \mathbb{A}^n$:

i) $I(Z(S)) \supseteq S$.

Pf/ Let $f \in S$ and $p \in Z(S)$. Thus, $g(p) = 0$ $\forall g \in S$. In particular, $f(p) = 0$. Hence, $f \in I(Z(S))$. $\square$

ii) $Z(I(X)) \supseteq X$.

Pf/ Let $p \in X$ and $f \in I(X)$. Then $f(q) = 0$ $\forall q \in X$. In particular, $f(p) = X$. Hence, $p \in Z(I(X))$. $\square$

2. a) If X, Y are algebraic sets, then $X = Y$ iff $I(X) = I(Y)$.

Pf/ Clearly, $X = Y \Rightarrow I(X) = I(Y)$. For the converse suppose $I(X) = I(Y)$. Since X, Y are algebraic, we may write $X = Z(J)$ and $Y = Z(K)$ for some ideals $J, K \subseteq R$. But then $I(Z(J)) = I(Z(K)) \Rightarrow$

$$X = Z(J) = Z(I(Z(J))) = Z(I(Z(K))) = Z(K) = Y. \square$$

b) $X \subseteq \mathbb{A}^n$ algebraic, $p \in \mathbb{A}^n - X$. Then $\exists f \in R$ s.t. $f(X) = 0$ and $f(p) = 1$.

Pf/ Let $Y = X \cup \{p\}$. By results appearing later in this assignment, we know $\{p\}$ is algebraic and finite unions of algebraic sets are algebraic, hence Y is algebraic. Then, by results appearing above, $X \subsetneq Y \Rightarrow I(X) \supsetneq I(Y)$

$\Longrightarrow \exists\, g \in I(X) \setminus I(Y)$. So $g(X) = 0$ but $g(p) \neq 0$. Define $f = g/g(p)$. $\square$

3. The following are algebraic sets.

a) $\{(t, t^2, t^3) : t \in k\}$.

Pf/ Let $R = k[x,y,z]$. Then $\{(t,t^2,t^3) : t \in k\} = Z(y - x^2, z - x^3)$. $\square$

b) $\{(\cos t, \sin t) : t \in \mathbb{R}\}$.

Pf/ Let $R = \mathbb{R}[x,y]$. Then $\{(\cos t, \sin t) : t \in \mathbb{R}\} = Z(x^2 + y^2 - 1)$. $\square$

4. (a) The radical of an ideal is an ideal.

Pf/ Let I be an ideal, and let $f, g \in \operatorname{rad}(I)$, $h \in R$. Then Take s, t such that $f^s, g^t \in I$. Then $(hf)^s = h^s f^s \in I$ since I is an ideal. Thus, $hf \in \operatorname{rad}(I)$. Then, to see $f + g \in \operatorname{rad}(I)$ note that $(f+g)^{s+t} = \sum_{i=0}^{s+t} a_i f^i g^{s+t-i}$ for some constants a_i. For each i in the sum, either $i \geq s$ or $s+t-i \geq t$. Hence, either $f^i \in I$ or $g^{s+t-i} \in I$. Either way,

$f^i g^{s+t-i} \in I$. Hence, $(f+g)^{s+t-i} \in I$. $\square$

(b) $I(X)$ is radical for all $X \subseteq \mathbb{A}^n$.

Pf/ Let $f^m \in I(X)$ for some $m > 0$ and let $p \in X$. Then $f^m(p) = (f(p))^m = 0$
$\Rightarrow f(p) = 0$ (since k is a field). Hence, $f \in I(X)$. $\square$

(c) Let $I \subseteq R$ be an ideal. Then I prime $\Rightarrow$ I radical.

Pf/ Suppose I is prime and $f^m \in I$ for some $m > 0$. Then $f(f^{m-1}) \in I$ and I prime $\Rightarrow f \in I$ or $f^{m-1} \in I$. Hence, by induction on m, $f \in I$. $\square$

(d) Let $J \subseteq R$ be an ideal. Then $Z(J) = Z(\text{rad}\, J)$ and $\text{rad}(J) \subseteq I(Z(J))$.

Pf/ $J \subseteq \text{rad}(J) \Rightarrow Z(J) \supseteq Z(\text{rad}(J))$. Conversely, let $p \in Z(J)$ and let $f \in \text{rad}(J)$. Then $f^m \in J$ for some $m > 0$. Thus, $f^m(p) = (f(p))^m = 0 \Rightarrow f(p) = 0$. So $p \in Z(\text{rad}(J))$. This shows $Z(J) = Z(\text{rad}(J))$.

Next, let $f \in \text{rad}(J)$ and $p \in Z(J)$. By what we have just shown, $p \in Z(\text{rad}(J))$, so $f(p) = 0$. Thus, $f \in I(Z(J))$. $\square$

5. (a) $\emptyset$ and $\mathbb{A}^n$ are algebraic sets.

Pf/ $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$. $\square$

(b) An arbitrary intersection of algebraic sets is algebraic.

Pf/ Let $X_\alpha = Z(J_\alpha) \subseteq \mathbb{A}^n$ for α in some index set A and $J_\alpha \subseteq R$ $\forall \alpha$.

Then $\bigcap_{\alpha \in A} X_\alpha = Z(\bigcup_{\alpha \in A} J_\alpha)$. $\square$

(c) A finite union of algebraic sets is algebraic.

Pf/ Let $X_i = Z(J_i)$ for $i = 1, 2$ with $J_1, J_2 \subseteq R$. Then

$X_1 \cup X_2 = Z(\{fg : f \in J_1, g \in J_2\})$. To see this, first let $p \in X_1 \cup X_2$.

Without loss of generality, take $p \in X_1$. Then let $f \in J_1$, $g \in J_2$. We have

$(fg)(p) = f(p)g(p) = 0 \cdot g(p) = 0$. Hence, $X_1 \cup X_2 \subseteq Z(\{fg : f \in J_1, g \in J_2\})$.

Conversely, suppose $p \in \mathbb{A}^n$ and $f(p)g(p) = 0$ $\forall f \in J_1, g \in J_2$. If $f(p) = 0$ $\forall$

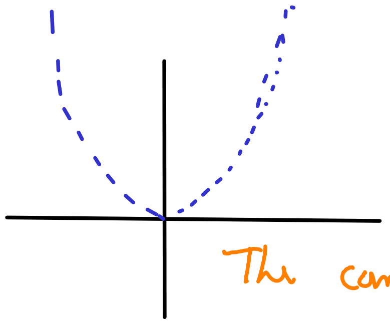
$f \in J_1$, then $p \in X_1 = Z(J_1)$, and hence, $p \in X_1 \cup X_2$. Otherwise, there exists $f \in J_1$ such that $f(p) \neq 0$. Let $g \in J_2$. Then $f(p)g(p) = 0$ and $f(p) \neq 0$ $\Rightarrow g(p) = 0$. Thus, $g(p) = 0$ $\forall g \in J_2$. So $p \in Z(J_2) = X_2 \subseteq X_1 \cup X_2$.

By induction, a finite union of algebraic sets is algebraic. $\square$

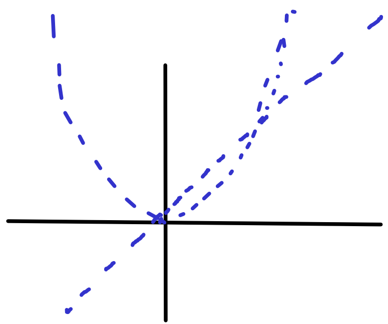
(d) Let $\tau = \{ U \subseteq \mathbb{A}^n : U^c \text{ is algebraic} \}$. Then τ is a topology on $\mathbb{A}^n$.

Pf/ By part (a), $\emptyset \in \tau$ and $\mathbb{A}^n \in \tau$. Let $U_\alpha \in \tau$ for $\alpha \in A$. Then $\left(\bigcup_{\alpha \in A} U_\alpha \right)^c = \bigcap_{\alpha \in A} U_\alpha^c$ is algebraic by part (b). Hence, τ is closed under arbitrary unions. Similarly, if $U_1, \ldots, U_s \in \tau$, then $(U_1 \cap \cdots \cap U_s)^c = U_1^c \cup \cdots \cup U_s^c$ is algebraic by (c). So τ is closed under finite intersections. $\square$

(e) Pictures of some open sets in $\mathbb{A}^2_{\mathbb{R}}$:



The complement of $y - x^2 = 0$.



The complement of $y - x^2 = 0$ union with the complement of $y - x = 0$.

(f) The Zariski topology is not Hausdorff in general.

Pf/ Consider $\mathbb{R} = \mathbb{A}^1_{\mathbb{R}}$ with the Zariski topology. Let U be an open set containing 0. Then $U^c = Z(I)$ for some ideal $I \subseteq \mathbb{R}[x]$. Since $\mathbb{R}[x]$ is a PID, we can write $I = (f)$ for some polynomial $f \in \mathbb{R}[x]$. Since $0 \notin U^c = Z(f)$, $f \neq 0$. So U^c is finite. Similarly, if 1 is in some open set V, then V^c is finite. Therefore, $(U \cap V)^c = U^c \cup V^c$ is a finite set. In particular, $U^c \cup V^c \neq \mathbb{A}^n$. Hence, $U \cap V \neq \emptyset$. $\square$

6. If $p \in \mathbb{A}^n$, then $\{p\}$ is an algebraic set. Every finite subset of $\mathbb{A}^n$ is algebraic.

Pf/ Let $p = (p_1, \dots, p_n)$. Then $\{p\} = Z(x_1 - p_1, \dots, x_n - p_n)$. By part (c) of the previous problem, the finite union of algebraic sets is an algebraic set. Hence, finite collections of points are algebraic sets. $\square$