Math 332 HW 8 solutions

1. (a) 3-Sylow subgroups of A_5.

Solution: $|A_5| = 5!/2 = 3 \cdot 20$. The subgroups of order 3 are

$\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(125)\rangle$, $\langle(134)\rangle$, $\langle(135)\rangle$, $\langle(145)\rangle$, $\langle(234)\rangle$,

$\langle(235)\rangle$, $\langle(245)\rangle$, $\langle(345)\rangle$. $\square$

So there are 10 3-Sylow subgroups. Note that $10 \mid 20$ and $10 \equiv 1 \bmod 3$.

(b) Every group of order 56 has a nontrivial normal subgroup.

Solution: $56 = 2^3 \cdot 7$. The number of 7-Sylow subgroups divides $2^3 = 8$ and is congruent to 1 mod 7. If there is just one, since all of its conjugates are 7-Sylow subgroups, we see it has only one conjugate and, hence, is normal. If there are 8 conjugates, then there are $8 \cdot 6 = 48$ elements of order 7 in the group. The remaining 8 elements must constitute a unique, hence normal, Sylow 2-group.

(c) How many 5-Sylow subgroups of S_5? Exhibit 2 of them. ②

Solution: $|S_5| = 5! = 5 \cdot 24$. The number of 5-Sylow subgroups divides 24 and is congruent to 1 mod 5: Divisors of 24: 1, 2, 3, 4, 6, 8, 12, 24. congruent to 1 mod 5. Thus, there is one or there are 6.

In fact, there are 6. They are all conjugate (as Sylow tells us). They are the 6 distinct groups generated by 5-cycles. Here are two:

$$\langle (12345) \rangle, \quad \langle (12354) \rangle. \quad \square$$

(d) We have $175 = 5^2 \cdot 7$. The number of 5-Sylow subgroups is $\equiv 1 \bmod 5$ and divides 7. Hence, there is only one N, which must be normal. Since, $|N| = 25$ is the square of a prime, N is abelian (see thm. in our text). The number of 7-Sylow subgroups is $\equiv 1 \bmod 7$ and divides 25. Again, there is only one, K, which must be normal. Since K has prime order, it is cyclic. We have $N \cap K = \{e\}$ since the order of any element in the intersection must divide 25 and divide 7. Thus, the group is a direct product of abelian groups, $N \times H$, hence abelian.

2. (a) $U(R)$ is a multiplicative group.

Solution: $1 \in U(R)$, and if $u, v \in U(R)$ then $u^{-1}, v^{-1} \in U(R)$ since $uu^{-1} = vv^{-1} = 1$. Also, $uv \in U(R)$ since $uv(v^{-1}u^{-1}) = 1$. $\square$

(b) $U(\oplus_{i=1}^n R) = \oplus_{i=1}^n U(R_i)$.

Solution: Let $u \in U(\oplus_{i=1}^n R)$. Then $u = (u_1, \dots, u_n)$ and $\exists\, v = (v_1, \dots, v_n) \in \oplus R_i$ such that $u \cdot v = (u_1v_1, \dots, u_nv_n) = 1 = (1, \dots, 1)$. Hence, $u_iv_i = 1$ $\forall i$. So $u_i \in U(R_i)$ $\forall i$, i.e., $u \in \oplus U(R_i)$. We've shown $U(\oplus R_i) \leq \oplus U(R_i)$. Conversely, suppose $u = (u_1, \dots, u_n) \in \oplus U(R_i)$. Then $\forall i$, $\exists v_i \in R_i$ s.t. $u_iv_i = 1$. But $v := (v_1, \dots, v_n)$ is the inverse for u. So $u \in U(\oplus R_i)$. $\square$

(c) $a, b \in R$, a a unit, $b^2 = 0$. Show $a+b$ is a unit.

Solution: $(a+b)(a^{-1} - ba^{-2}) = 1$. $\square$

Motivation: $\frac{1}{a+b} = \frac{1}{a(1+ba^{-1})} = \frac{1}{a}\left(1 - ba^{-1} + \underbrace{(ba^{-1})^2 + \cdots}_{=0}\right)$. (geometric series)

3. A noncommutative ring with exactly 16 elements.

Solution: One possibility is $M_2(\mathbb{Z}_2)$ = ring of 2×2 matrices with entries in $\mathbb{Z}_2$. $\square$

4. (a) $I \cap J$ is an ideal.

Solution: If $a, b \in I\cap J$ then $a+b \in I$ and $a+b \in J$ since I and J are ideals. Thus, $a+b \in I \cap J$. Let $r \in R$ and $a \in I \cap J$. Since I and J are ideals, ra and ar are in I and J, hence in $I \cap J$. $\square$

(b) $I+J$ is an ideal.

Solution: Let $a_1, a_2 \in I+J$.
Say $a_1 = i_1 + j_1$, $a_2 = i_2 + j_2$ with $i_1, i_2 \in I$ and $j_1, j_2 \in J$. Then $a_1 + a_2 = (i_1+i_2) + (j_1+j_2) \in I+J$ since $i_1+i_2 \in I$ and $j_1+j_2 \in J$. If $r \in R$, then $ri_1, i_1 r \in I$ and $rj_1, rj_2 \in J$, so $r(i_1+j_1) = ri_1 + rj_1 \in I+J$ and $(i_1+j_1)r = i_1 r + j_1 r \in I+J$. $\square$

4. (c) Find ideals $I, J \in \mathbb{Q}[x,y]$ s.t. $\{ij : i \in I, j \in J\}$ is not an ideal.

Solution: For example, let $I = (x,y)$, $J = (x+y, x-y)$. Then

$$x(x+y) - y(x-y) = x^2+y^2 \in IJ \quad \text{but not in} \quad X := \{ij : i \in I, j \in J\}.$$

For sake of contradiction, suppose $x^2+y^2 \in X$. Then $\exists\, f,g,h,k \in \mathbb{Q}[x,y]$ such that $(xf+yg)((x+y)h+(x-y)k) = x^2+y^2$. This forces

$$x^2(fh-fk) + y^2(gh-gk) + xy(fh-fk+gh+gk) = x^2+y^2.$$

So $f(h-k) = g(h-k) = 1$ and $fh-fk+gh+gk = 0$. Now $f(h-k) = g(h-k) = 1$ $\Rightarrow f = g = 1$, $h = k$. Then $fh - fk + gh - gk = 0 \xRightarrow{\text{setting } f=g=1} 2h = 0 \Rightarrow h = 0 \xRightarrow{\text{since } h=k} k = 0$

But then $(xf+yg)((x+y)h+(x-y)k) = 0 \neq x^2+y^2$. $\square$

(d) R commutative, $I+J = R$. Show $IJ = I \cap J$.

Pf/ $IJ \subseteq I \cap J$ Let $i \in I$, $j \in J$. Then since I, J are ideals, $ij \in I \cap J$. Since the generators of IJ are in $I \cap J$, we have $IJ \subseteq I \cap J$.

$I \cap J \subseteq IJ$ Since $I+J=R$, $\exists\, i\in I, j\in J$ s.t. $i+j=1$. Say $a \in I\cap J$.
Then $a = a\cdot 1 = a(i+j) = ai + aj \in IJ$ since $ai, aj \in IJ$. For example, to see $ai \in IJ$, note $i \in I$ and $a \in J$. $\square$

5. R commutative, $X \subseteq R$, $Ann(X) = \{r \in R : rx = 0 \;\; \forall x \in X\}$. Prove $Ann(X)$ is an ideal.

Pf/ Suppose $r, s \in Ann(X)$. Then $(r+s)x = rx + sx = 0+0=0$, $\forall x \in X$.
So $r+s \in Ann(X)$. Now suppose $r \in Ann(X)$ and $s \in R$.
Then $(sr)x = s(rx) = s\cdot 0 = 0 \;\; \forall x \in X \Rightarrow sr \in Ann(X)$. $\square$