HW 8, due Tuesday, March 31 *Math 332*

1. Read up on the Sylow theorems (the wikipedia is fine).
   (a) Find all Sylow 3-subgroups of A_5.
   (b) Show that every group of order 56 has a proper nontrivial normal subgroup.
   (c) How many Sylow 5-subgroups of S_5 are there. Exhibit two.
   (d) Prove that every group of order 175 is abelian. (Theorems 22.11 and 23.3 are useful, in addition to the Sylow theorems.)
2. For a commutative ring with unity, R, let $U(R)$ be the set of units in R. Suppose R and $R_1, \ldots, R_n$ are commutative rings with unity.
   (a) Prove that $U(R)$ is a multiplicative group.
   (b) If $R_1, \ldots, R_n$ are commutative rings with unity, show that
   $$U(R_1 \oplus \cdots \oplus R_n) \approx U(R_1) \oplus \cdots \oplus U(R_n).$$
   (c) If $a, b \in R$, with a a unit and $b^2 = 0$, prove that $a + b$ is a unit.
3. Describe a noncommutative ring with exactly 16 elements.
4. Let I and J be ideals in a ring R. (You may assume R is commutative with unity, although that is not necessary.)
   (a) Prove that $I \cap J$ is an ideal.
   (b) Prove that $I + J := \{i + j : i \in I, j \in J\}$ is an ideal.
   (c) The product IJ is the ideal generated by all products ij with $i \in I$ and $j \in J$. Give examples of ideals $I, J \subset \mathbb{Q}[x, y]$ for which $\{ij \colon i \in I, j \in J\}$ is not an ideal. (Proof required.)
   (d) If R is a commutative ring with unity and $I + J = R$, prove that $I \cap J = IJ$.
5. Let R be a commutative ring, and let X be any subset of R. The *annihilator* of X is $\operatorname{Ann}(X) := \{r \in R : rx = 0 \text{ for all } x \in X\}$. Show that it is an ideal.
6. This problem was removed.