

Math 332 HW 7 solutions

①

1. (a) $eg\bar{e}^{-1} = g$ and $(h_1 h_2) g (h_1 h_2)^{-1} = h_1 (h_2 g h_2^{-1}) h_1^{-1}$

(b) $\bar{e}^{-1} g e = g$, but $(h_1 h_2)^{-1} g (h_1 h_2) = h_2^{-1} (h_1^{-1} g h_1) h_2 \neq h_1^{-1} (h_2^{-1} g h_2) h_1$, in general.

For example, let $G = S_3 = \langle \rho, \phi : \rho^3 = \phi^2 = 1, \rho\phi = \phi\rho^2 = 1 \rangle$. Then

$$(\rho\phi)^{-1} \phi (\rho\phi) = \phi \rho^2 \phi \rho \phi = \rho^2 \phi$$

which is not the same as

$$\rho^{-1} (\phi^{-1} \phi \phi) \rho = \rho^2 \phi \phi \phi \rho = \rho \phi.$$

(c) For each h , the mapping $c_h: G \rightarrow G$ is a homomorphism:
 $g \mapsto h g h^{-1}$

$$c_h(g_1 g_2) = h g_1 g_2 h^{-1} = (h g_1 h^{-1}) (h g_2 h^{-1}) = c_h(g_1) c_h(g_2).$$

Since h is arbitrary, we see $c_{h^{-1}}: g \mapsto h^{-1} g h$ is also a homomorphism.

To see they are automorphisms, note that c_h and $c_{h^{-1}}$ are inverses.

2. (a) N has index 2 in D_n ; hence $N \triangleleft D_n$.

(b) $1 \rightarrow N \xrightarrow{\gamma} D_n \xrightarrow{\Psi} H \rightarrow 1$ where γ is the inclusion and Ψ is the mapping $\Psi(\rho^i \rho^j) = \phi^j$. Define a splitting by $\begin{cases} H \rightarrow D_n \\ \phi \mapsto \phi \end{cases}$.

(c) $H \rightarrow \text{Aut}(N)$ where $\alpha(h) : N \rightarrow N$. So $\alpha(1)(\rho^i) = \rho^i$
 $h \mapsto \alpha(h)$ $\rho \mapsto h \rho h^{-1}$ $\alpha(\phi)(\rho^i) = \rho^{-i}$.

(d) Examples:

$$(\rho^i, \phi)(\rho^j, \phi) = (\rho^i \alpha(\phi)(\rho^j), \phi^2) = (\rho^{i-j}, 1)$$

$$(\rho^i, 1)(\rho^j, \phi) = (\rho^i \alpha(1)(\rho^j), \phi) = (\rho^{i+j}, \phi)$$

$$(\rho^i, \phi)(\rho^j, 1) = (\rho^i \alpha(\phi)\rho^j, \phi) = (\rho^{i-j}, \phi).$$

$$\text{So } (\rho, 1)(1, \phi) = (\rho, \phi) \text{ and } (1, \phi)(\rho, 1) = (\rho^{-1}, \phi) = (\rho^{n-1}, \phi).$$

(e) The proper subgroups of D_4 are abelian, and hence so are direct products of these subgroups. But D_4 is not abelian.

3. If $j: H \rightarrow G$ is a splitting, define $\begin{cases} G \xrightarrow{\varepsilon} N \rtimes H \\ g \mapsto (g [j(\psi(g))]^{-1}, \psi(g)) \end{cases}$. ③

We check it's a homomorphism:

$$gh \rightarrow (gh \ j(\psi(gh))^{-1}, \psi(gh))$$

and

$$\begin{aligned} & (g \ j(\psi(g))^{-1}, \psi(g)) (h \ j(\psi(h))^{-1}, \psi(h)) \\ &= (g \ j(\psi(g))^{-1}, \psi(g)) [h \ j(\psi(h))^{-1} j(\psi(g))^{-1}, \psi(g)\psi(h)] \\ &= (gh \ (j(\psi(g)) j(\psi(h)))^{-1}, \psi(gh)) \\ &= (gh \ j(\psi(gh))^{-1}, \psi(gh)). \quad \checkmark \end{aligned}$$

For the inverse, define $N \rtimes H \xrightarrow{\delta} G$
 $(n, h) \mapsto n j(h)$

For ease of notation,
 consider $N \triangleleft G$.

To see it's a homomorphism, note that $(n_1, h_1)(n_2, h_2) = (n_1 j(h_1) n_2 j(h_1)^{-1}, h_1 h_2)$
 $\rightarrow n_1 j(h_1) n_2 j(h_1)^{-1} j(h_1 h_2) = [n_1 j(h_1)] [n_2 j(h_2)]. \quad \checkmark$

Check ε, δ are inverses:

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$$\varepsilon(\delta(n, h)) = \varepsilon(n_j(h)) = (n_j(h) [j(\Psi(n_j(h)))]^{-1}, \Psi(n_j(h)))$$

$$= (n_j(h) [j(\Psi(n) \circ \Psi_j) h]^{-1}, \Psi(n) \circ \Psi_j(h))$$

$$= (n_j(h) [j(\Psi(n)) j(h)]^{-1}, \Psi(n) h)$$

(since $\Psi_j = \text{id}_H$)

$$= (n_j(h) j(h)^{-1}, h)$$

(since $n \in \ker \Psi$)

$$= (n, h),$$

and

$$\delta(\varepsilon(g)) = \delta(g j(\Psi(g))^{-1}, \Psi(g)) = g j(\Psi(g))^{-1} j(\Psi(g)) = g. \quad \square$$

$$4. (a) \quad 1 \triangleleft \langle p^2 \rangle \triangleleft \langle p \rangle \triangleleft D_4 \quad (p = \text{rotation} = (1234))$$

$$\text{factors: } \langle p^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}, \quad \langle p \rangle / \langle p^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}, \quad D_4 / \langle p \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

$$(b) \quad 1 \triangleleft \langle (13)(24) \rangle \triangleleft \langle (13)(24), (14)(23) \rangle \triangleleft A_4 \triangleleft S_4$$

$$\text{factors: } \langle (13)(24) \rangle \cong \mathbb{Z}/2\mathbb{Z}, \quad \langle (13)(24), (14)(23) \rangle / \langle (13)(24) \rangle \cong \mathbb{Z}/2\mathbb{Z}, \quad \frac{A_4}{\langle (13)(24), (14)(23) \rangle} \cong \mathbb{Z}/3\mathbb{Z}, \quad S_4 / A_4 \cong \mathbb{Z}/2\mathbb{Z}.$$

(c) $1 \triangleleft A_n \triangleleft S_n$

factors: $A_n, S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$

(d) $1 \triangleleft \mathbb{Z}/32 \times \{0\} \triangleleft \mathbb{Z}/32 \times \mathbb{Z}/2 \triangleleft \mathbb{Z}/32 \times \mathbb{Z}/6 \triangleleft \mathbb{Z}/32 \times \mathbb{Z}/12 \triangleleft \mathbb{Z}/32 \times \mathbb{Z}/24$

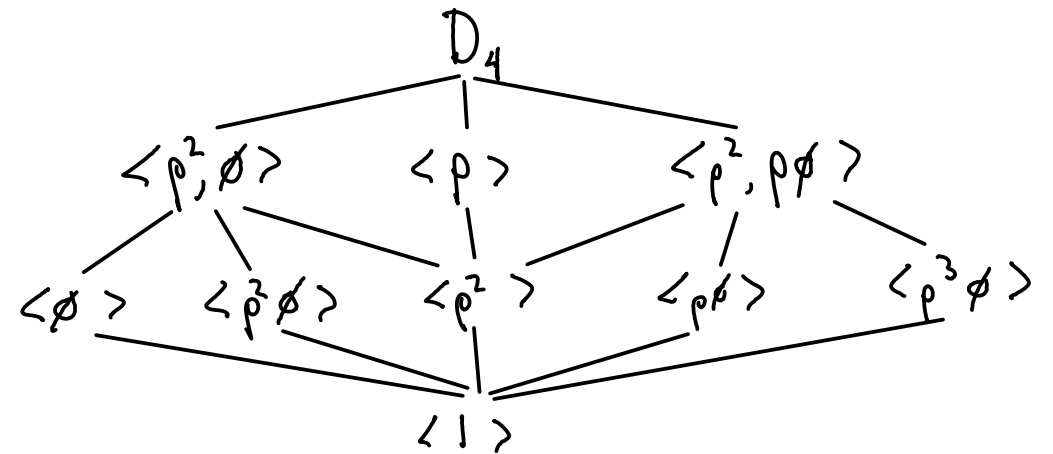
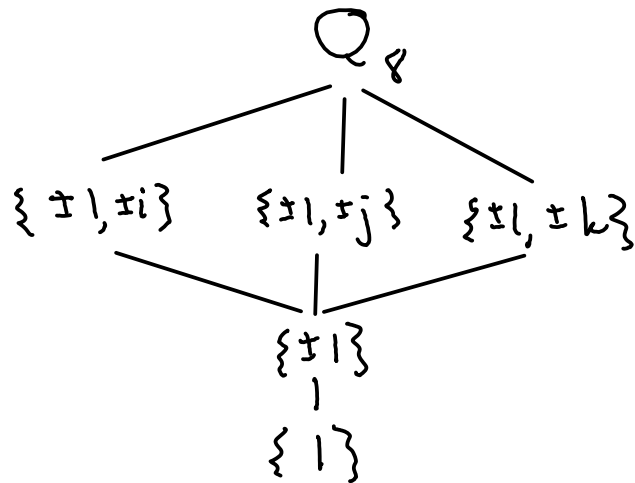
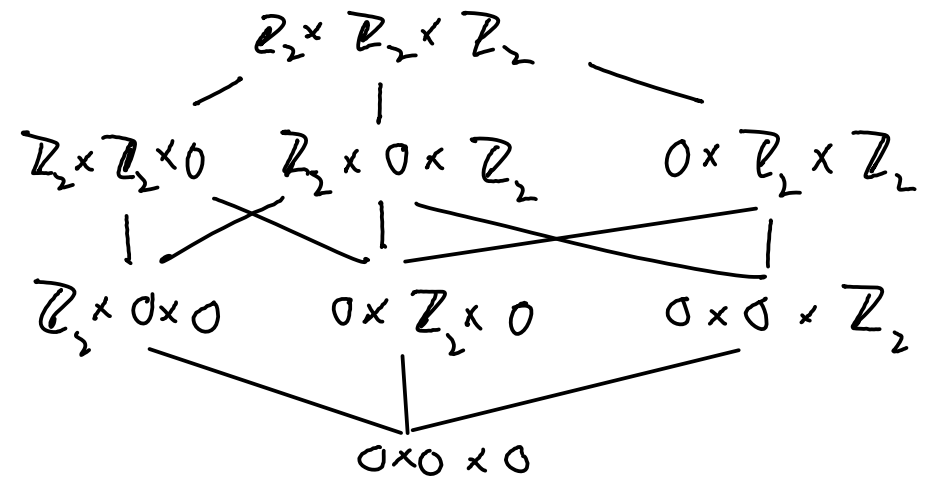
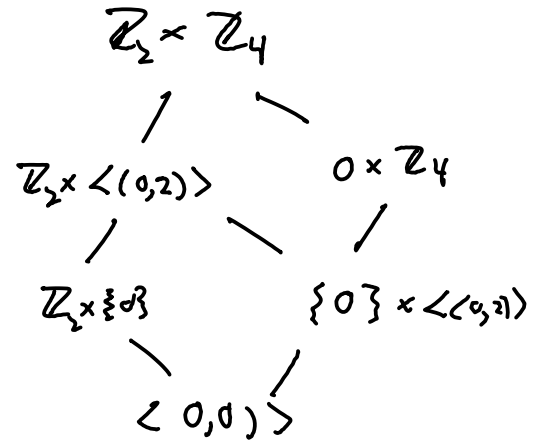
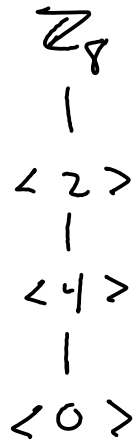
factors $\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{Z}/32 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/3\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{matrix}$

5 (a)

	1	-1	i	j	k	-i	-j	-k
1	1	-1	i	j	k	-i	-j	-k
-1	-1	1	-i	-j	-k	i	j	k
i	i	-i	-1	k	-j	1	-k	j
j	j	-j	-k	-1	i	k	1	-i
k	k	-k	j	-i	-1	-j	i	1
-i	-i	i	1	-k	j	-1	k	-j
-j	-j	j	k	1	-i	-k	-1	i
-k	-k	k	-j	i	1	j	-i	-1

(b) $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, Q_8 , D_4

(6)



(c) All of the subgroups of Q_8 are normal in Q_8 .

1 and Q_8 are trivially normal

$\{\pm 1, \pm i\}$ is normal since $\pm_j \{\pm 1, \pm i\} = \{\pm_j, \pm k\} = \{\pm 1, \pm i\}(\pm_j)$, and

$\pm_k \{\pm 1, \pm i\} = \{\pm_k, \pm j\} = \{\pm 1, \pm i\}(\pm_k)$.

Similarly, $\{\pm 1, \pm j\}$ and $\{\pm 1, \pm k\}$ are normal.

The subgroup $\{\pm 1\}$ is normal since it's in the center.

(d) (i) $1 \triangleleft \{\pm 1\} \triangleleft \{\pm 1, \pm i\} \triangleleft Q_8$

(ii) $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$

$\frac{\{\pm 1, \pm i\}}{\langle \pm 1 \rangle} \cong \mathbb{Z}/2\mathbb{Z}$

$Q_8 / \langle \pm 1, \pm i \rangle \cong \mathbb{Z}/2\mathbb{Z}$

(e) Q_8 is not the semidirect product of any of its proper subgroups. To

see this, note that its proper subgroups are isomorphic to \mathbb{Z}_2 and \mathbb{Z}_4 .

A semidirect product of the form $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ corresponds to a homomorphism $\mathbb{Z}_2 \rightarrow \text{Aut}(\mathbb{Z}_4)$. We saw in class that there are two possibilities: $\mathbb{Z}_2 \times \mathbb{Z}_4$ and D_4 . (And Q_8 is not isomorphic to either of these, as can be seen from the subgroup lattices.)

A semidirect product of the form $\mathbb{Z}_2 \rtimes \mathbb{Z}_4$ corresponds to a homomorphism $\mathbb{Z}_4 \rightarrow \text{Aut}(\mathbb{Z}_2)$. There is only one such homomorphism, sending everything in \mathbb{Z}_4 to the identity mapping $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2$. This gives $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Alternate solution: Note that non-trivial subgroups all contain ± 1 . If Q_8 were a semi-direct product of two proper subgroups then it would be an internal semidirect product. This means there would be proper subgroups $N, H < Q_8$ with $N \triangleleft Q_8$, $NH = Q_8$, and $N \cap H = \{1\}$.

↪ Not possible in our case.