Math 332 HW 7 solutions

1. (a) $ege^{-1} = g$ and $(h_1h_2)g(h_1h_2)^{-1} = h_1(h_2gh_2^{-1})h_1^{-1}$

(b) $e^{-1}ge = g$, but $(h_1h_2)^{-1}g(h_1h_2) = h_2^{-1}(h_1^{-1}gh_1)h_2 \neq h_1^{-1}(h_2^{-1}gh_2)h_1$, in general.

For example, let $G = S_3 = \langle \rho, \phi : \rho^3 = \phi^2 = 1, \rho\phi = \phi\rho^2 = 1 \rangle$. Then

$$(\rho\phi)^{-1}\phi(\rho\phi) = \phi\rho^2\phi\rho\phi = \rho^2\phi$$

which is not the same as

$$\rho^{-1}(\phi^{-1}\phi\phi)\rho = \rho^2\phi\phi\phi\rho = \rho\phi.$$

(c) For each h, the mapping $c_h : G \to G$, $g \mapsto hgh^{-1}$ is a homomorphism:

$$c_h(g_1g_2) = hg_1g_2h^{-1} = (hg_1h^{-1})(hg_2)h^{-1} = c_h(g_1)c_h(g_2).$$

Since h is arbitrary, we see $c_{h^{-1}} : g \mapsto h^{-1}gh$ is also a homomorphism.

To see they are automorphisms, note that c_h and $c_{h^{-1}}$ are inverses.

2. (a) N has index 2 in D_n; hence $N \triangleleft D_n$.

(b) $1 \to N \xrightarrow{\gamma} D_n \xrightarrow{\psi} H \to 1$ where γ is the inclusion and ψ is the mapping $\psi(\rho^i \phi^j) = \phi^j$. Define a splitting by $\left\{ \begin{array}{l} H \to D_n \\ \phi \mapsto \phi \end{array} \right.$.

(c) $\begin{array}{l} H \to \mathrm{Aut}(N) \\ h \mapsto \alpha(h) \end{array}$ where $\begin{array}{l} \alpha(h) : N \longrightarrow N \\ \quad\quad \rho \mapsto h \rho h^{-1} \end{array}$. So $\begin{array}{l} \alpha(1)(\rho^i) = \rho^i \\ \alpha(\phi)(\rho^i) = \rho^{-i} \end{array}$.

(d) Examples:

$$(\rho^i, \phi)(\rho^j, \phi) = (\rho^i \alpha(\phi)(\rho^j), \phi^2) = (\rho^{i-j}, 1)$$

$$(\rho^i, 1)(\rho^j, \phi) = (\rho^i \alpha(1)(\rho^j), \phi) = (\rho^{i+j}, \phi)$$

$$(\rho^i, \phi)(\rho^j, 1) = (\rho^i \alpha(\phi)\rho^j, \phi) = (\rho^{i-j}, \phi).$$

So $(\rho, 1)(1, \phi) = (\rho, \phi)$ and $(1, \phi)(\rho, 1) = (\rho^{-1}, \phi) = (\rho^{n-1}, \phi)$.

(e) The proper subgroups of D_4 are abelian, and hence so are direct products of these subgroups. But D_4 is not abelian.

3. If $j: H \to G$ is a splitting, define $\begin{cases} G \xrightarrow{\varepsilon} N \rtimes H \\ g \mapsto (g\,[j(\psi(g))]^{-1}, \psi(g)) \end{cases}$. ③

We check it's a homomorphism:

$$gh \to ((gh)\ j(\psi(gh))^{-1}, \psi(gh))$$

and

$$(g\,j(\psi(g))^{-1}, \psi(g))(h\ j(\psi(h))^{-1}, \psi(h))$$

$$= (g\ j(\psi(g))^{-1} j\psi(g)\,[h\,j(\psi(h))^{-1} j(\psi(g))^{-1}, \psi(g)\psi(h))$$

$$= (gh\ (j(\psi(g))\,j(\psi(h)))^{-1}, \psi(gh))$$

$$= (gh\ j\psi(gh)^{-1}, \psi(gh)). \checkmark$$

For the inverse, define $\begin{matrix} N \rtimes H \xrightarrow{\delta} G \\ (n, h) \mapsto nj(h) \end{matrix}$

For ease of notation, consider $N \trianglelefteq G$.

To see it's a homomorphism, note that $(n_1, h_1)(n_2, h_2) = (n_1\ j(h_1) n_2\ j(h_1)^{-1}, h_1 h_2)$

$$\to n_1 j(h_1) n_2 j(h_1)^{-1} j(h_1 h_2) = [n_1 j(h_1)][n_2\ j(h_2)]. \checkmark$$

Check ε, δ are inverses:

$$\varepsilon(\delta(n,h)) = \varepsilon(n\,j(h)) = \left(n\,j(h)\left[j(\psi(n\,j(h)))\right]^{-1}, \psi(n\,j(h))\right)$$

$$= \left(n\,j(h)\left[j\left(\psi(n)(\psi\circ j)h\right)\right]^{-1}, \psi(n)(\psi\circ j)(h)\right)$$

$$= \left(n\,j(h)\left[j(\psi(n))j(h)\right]^{-1}, \psi(n)\,h\right) \quad (\text{since } \psi\circ j = id_H)$$

$$= \left(n\,j(h)\,j(h)^{-1}, h\right) \quad (\text{since } n \in \ker\psi)$$

$$= (n, h),$$

and

$$\delta(\varepsilon(g)) = \delta\left(g\,j(\psi(g))^{-1}, \psi(g)\right) = g\,j(\psi(g))^{-1}\,j(\psi(g)) = g. \quad \square$$

4. (a) $1 \triangleleft \langle \rho^2 \rangle \triangleleft \langle \rho \rangle \triangleleft D_4 \qquad (\rho = \text{rotation} = (1234))$

factors: $\langle \rho^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$, $\langle \rho \rangle / \langle \rho^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$, $D_4/\langle \rho \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

(b) $1 \triangleleft \langle (13)(24) \rangle \triangleleft \langle (13)(24), (14)(23) \rangle \triangleleft A_4 \triangleleft S_4$

factors: $\langle (13)(24) \rangle \cong \mathbb{Z}/2\mathbb{Z}$, $\langle (13)(24), (14)(23) \rangle / \langle (13)(24) \rangle \cong \mathbb{Z}/2\mathbb{Z}$, $A_4 / \langle (13)(24), (14)(23) \rangle \cong \mathbb{Z}/3\mathbb{Z}$, $S_4/A_4 \cong \mathbb{Z}/2\mathbb{Z}$.

(c) $1 \triangleleft A_n \triangleleft S_n$

factors: A_n, $S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$

(d) $1 \triangleleft \mathbb{Z}/3\mathbb{Z} \times \{0\} \triangleleft \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \triangleleft \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \triangleleft \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$

factors $\mathbb{Z}/3\mathbb{Z}$ $\quad$ $\mathbb{Z}/2\mathbb{Z}$ $\quad$ $\mathbb{Z}/3\mathbb{Z}$ $\quad$ $\mathbb{Z}/2\mathbb{Z}$

5 (a)

	1	-1	i	j	k	-i	-j	-k
1	1	-1	i	j	k	-i	-j	-k
-1	-1	1	-i	-j	-k	i	j	k
i	i	-i	-1	k	-j	1	-k	j
j	j	-j	-k	-1	i	k	1	-i
k	k	-k	j	-i	-1	-j	i	1
-i	-i	i	1	-k	j	-1	k	-j
-j	-j	j	k	1	-i	-k	-1	i
-k	-k	k	-j	i	1	j	-i	-1

(b) $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, Q_8, D_4

$$
\begin{tikzcd}
\mathbb{Z}_8 \arrow[d, no head] \\
\langle 2 \rangle \arrow[d, no head] \\
\langle 4 \rangle \arrow[d, no head] \\
\langle 0 \rangle
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{Z}_2 \times \mathbb{Z}_4 \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Z}_2 \times \langle (0,2) \rangle \arrow[d, no head] \arrow[drr, no head] & & 0 \times \mathbb{Z}_4 \arrow[d, no head] \\
\mathbb{Z}_2 \times \{0\} \arrow[dr, no head] & & \{0\} \times \langle (0,2) \rangle \arrow[dl, no head] \\
& \langle (0,0) \rangle &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times 0 \arrow[d, no head] \arrow[dr, no head] & \mathbb{Z}_2 \times 0 \times \mathbb{Z}_2 \arrow[dl, no head] \arrow[dr, no head] & 0 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[dl, no head] \arrow[d, no head] \\
\mathbb{Z}_2 \times 0 \times 0 \arrow[dr, no head] & 0 \times \mathbb{Z}_2 \times 0 \arrow[d, no head] & 0 \times 0 \times \mathbb{Z}_2 \arrow[dl, no head] \\
& 0 \times 0 \times 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& Q_8 \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
\{\pm 1, \pm i\} \arrow[dr, no head] & \{\pm 1, \pm j\} \arrow[d, no head] & \{\pm 1, \pm k\} \arrow[dl, no head] \\
& \{\pm 1\} \arrow[d, no head] & \\
& \{1\} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & D_4 \arrow[dll, no head] \arrow[d, no head] \arrow[drr, no head] & & \\
\langle \rho^2, \phi \rangle \arrow[d, no head] \arrow[dr, no head] \arrow[drr, no head] & & \langle \rho \rangle \arrow[d, no head] & & \langle \rho^2, \rho\phi \rangle \arrow[dll, no head] \arrow[dl, no head] \arrow[d, no head] \\
\langle \phi \rangle \arrow[drr, no head] & \langle \rho^2\phi \rangle \arrow[dr, no head] & \langle \rho^2 \rangle \arrow[d, no head] & \langle \rho\phi \rangle \arrow[dl, no head] & \langle \rho^3\phi \rangle \arrow[dll, no head] \\
& & \langle 1 \rangle & &
\end{tikzcd}
$$

(c) All of the subgroups of Q_8 are normal in Q_8.

1 and Q_8 are trivially normal

$\{\pm 1, \pm i\}$ is normal since $\pm j\{\pm 1, \pm i\} = \{\pm j, \pm k\} = \{\pm 1, \pm i\}(\pm j)$, and

$\pm k\{\pm 1, \pm i\} = \{\pm k, \pm j\} = \{\pm 1, \pm i\}(\pm k)$.

Similarly, $\{\pm 1, \pm j\}$ and $\{\pm 1, \pm k\}$ are normal.

The subgroup $\{\pm 1\}$ is normal since it's in the center.

(d) (i) $1 \lhd \{\pm 1\} \lhd \{\pm 1, \pm i\} \lhd Q_8$

(ii) $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ $\qquad$ $\dfrac{\{\pm 1, \pm i\}}{\langle \pm 1 \rangle} \cong \dfrac{\mathbb{Z}}{2\mathbb{Z}}$ $\qquad$ $Q_8 / \{\pm 1, \pm i\} \cong \mathbb{Z}/2\mathbb{Z}$

(e) Q_8 is not the semidirect product of any of its proper subgroups. To see this, note that its proper subgroups are isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_4$.

A semidirect product of the form $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ corresponds to a homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_4)$. We saw in class that there are two possibilities: $\mathbb{Z}_2 \times \mathbb{Z}_4$ and D_4. (And Q_8 is not isomorphic to either of these, as can be seen from the subgroup lattices.)

A semidirect product of the form $\mathbb{Z}_2 \rtimes \mathbb{Z}_4$ corresponds to a homomorphism $\mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_2)$. There is only one such homomorphism, sending everything in $\mathbb{Z}_4$ to the identity mapping $\mathbb{Z}_2 \to \mathbb{Z}_2$. This gives $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Alternate solution: Note that non-trivial subgroups all contain ± 1. If Q_8 where a semi-direct product of two proper subgroups then it would be an internal semidirect product. This means there would be proper subgroups $N, H < Q_8$ with $N \trianglelefteq Q_8$, $NH = Q_8$, and $N \cap H = \{1\}$.

Not possible in our case.