HW 7, due Tuesday, March 24 *Math 332*

1. Let G be a group.

   (a) Show that
   $$\begin{array}{rcl} G \times G & \to & G \\ (h,g) & \mapsto & hgh^{-1} \end{array}$$
   defines an action of G on itself.

   (b) Show that
   $$\begin{array}{rcl} G \times G & \to & G \\ (h,g) & \mapsto & h^{-1}gh \end{array}$$
   does not define an action of G on itself, in general. Give an concrete counterexample.

   (c) Fix $h \in G$. Show that $g \mapsto hgh^{-1}$ and $g \mapsto h^{-1}gh$ both define automorphisms of G.

2. Consider the dihedral group $D_n = \langle \rho, \phi : \rho^n = \phi^2 = 1, \rho\phi = \phi\rho^{n-1} \rangle$. Let $N = \langle \rho \rangle$ and $H = \langle \phi \rangle$.

   (a) Prove that N is a normal subgroup of D_n. (There is a really quick solution to this one!)

   (b) Show that $D_n = N \rtimes H$ by exhibiting a split exact sequence.

   (c) Describe the corresponding homomorphism $H \to \mathrm{Aut}(N)$.

   (d) Compute the products $(\rho^i, \phi)(\rho^j, \phi)$, $(\rho^i, 1)(\rho^j, \phi)$, $(\rho^i, \phi)(\rho^j, 1)$ in $N \rtimes H$. As a special case, compute $(\rho, 1)(1, \phi)$ and compare it to $(1, \phi)(\rho, 1)$.

   (e) Prove that D_4 is not the direct product of two of its proper subgroups.

3. Prove that if
   $$1 \to N \xrightarrow{\phi} G \xrightarrow{\psi} H \to 1$$
   is split exact, then $G \approx N \rtimes H$. (Hint: if $j \colon H \to G$ is a splitting, then
   $$g \cdot j(\psi(g))^{-1} \in\ N.)$$

4. Exhibit a composition series and composition factors for the following groups. Identify the composition factors as well-known groups.

   (a) D_4

   (b) S_4

   (c) S_n for $n \geq 5$

   (d) $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.

   Feel free to use Sage. You do not have to prove your results.

5. Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. The multiplication is determined by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Multiplication by -1 works as expected.

   (a) Make a multiplication table for Q_8.

   (b) Describe the five groups of order 8 and construct their subgroup lattices.

   (c) Decide which subgroups of Q_8 are or are not normal (with proof).

   (d) (i) Give a composition series for Q_8. (ii) What are the composition factors?

   (e) Is Q_8 a semidirect product of any of its proper subgroups (with proof)?