Math 332 HW 6 solutions

21.3 $SL(n,\mathbb{R}) \trianglelefteq GL(n,\mathbb{R})$

Pf/ Let $A \in SL(n,\mathbb{R})$ and $M \in GL(n,\mathbb{R})$. Then $\det(MAM^{-1}) =$ $\det M \det A (\det M)^{-1} = \det A = 1 \Rightarrow MAM^{-1} \in SL(n,\mathbb{R})$. $\square$

21.4. Let $\varphi : G \to H$ be a homomorphism of groups. Then $\ker\varphi \trianglelefteq G$.

Pf/ Let $a \in \ker\varphi$ and $g \in G$. Then $\varphi(gag^{-1}) = \varphi(g)\varphi(a)\varphi(g)^{-1}$ $= \varphi(g)\cdot 1_H \cdot \varphi(g)^{-1} = \varphi(g)\varphi(g)^{-1} = 1 \Rightarrow gag^{-1} \in \ker\varphi$. $\square$

21.5 If H is a subgroup of G and $[G:H] \leq 2$, then $H \trianglelefteq G$.

Pf/ If $[G:H] = 1$, then $H = G$, and $G \trianglelefteq G$. So suppose $[G:H] = 2$. Choose $x \notin H$, then G is partitioned into cosets H and xH. However, G is also partitioned into cosets H and Hx. So $G \setminus H = xH$ and $G \setminus H = Hx$. Hence, $xH = Hx$, as required. $\square$

22.8 Let G be a group and p a prime. Show $|G|$ is a power of p iff every element of G has order a power of p.

Pf/ ($\Rightarrow$) Suppose $|G|$ is a power of p and $x \in G$. Then $|x| \,\big|\, |G|$ by Lagrange. So $|x|$ is a power of p.

($\Leftarrow$) If $|G|$ is not a power of p, $\exists$ another prime q dividing $|G|$. By Cauchy's thm. (Thm. 22.7), G has an element of order q. $\square$

24.4 $H \trianglelefteq G$ with $[G:H] = n \Rightarrow a^n \in H \quad \forall a \in G$.

Give an example of a non-normal subgroup for which this fails.

Solution/ First, we are given that the group G/H has order n. Thus, given $a \in G$, the element $aH \in G/H$ has order dividing $|G/H| = n$ by Lagrange. So $(aH)^n = a^nH = H$, which implies $a^n \in H$ (since $1 \in H$). $\square$

For an example showing the result does not hold for non-normal subgroups in general. Let $G = D_6 = \langle \rho, \phi : \rho^6 = \phi^2 = 1,\ \rho\phi = \rho^5\phi \rangle$, and let $H = \langle \rho^3, \phi \rangle = \{1, \rho^3, \phi, \rho^3\phi\}$. Then $[G:H] = 3$ and $(\rho\phi)^3 = \rho\phi \notin H$. $\square$

1. $S_3 = \{(), (12), (13), (23), (123), (132)\}.$

$\phantom{S_3 = \{}1 \quad\;\; \phi \quad\;\; \rho\phi \quad\;\; \rho^2\phi \quad\;\; \rho \quad\;\; \rho^2$

$\rho\phi = \phi\rho^2, \quad \rho^3 = \phi^2 = 1$

Subgroups

$\langle 1 \rangle$ normal $\quad S_3/\langle 1 \rangle = S_3$

$\langle \phi \rangle$ not normal $\quad \rho\{1,\phi\} = \{\rho, \rho\phi\} \neq \{\rho, \rho^2\phi\} = \{1,\phi\}\rho$

$\langle \rho\phi \rangle$ not normal $\quad \phi\{1,\rho\phi\} = \{\phi, \rho^2\} \neq \{\phi, \rho\} = \{1,\rho\phi\}\phi$

$\langle \rho^2\phi \rangle$ not normal $\quad \rho\{1,\rho^2\phi\} = \{\rho, \phi\} \neq \{\rho, \rho\phi\} = \{1,\rho^2\phi\}\rho$

$\langle \rho \rangle$ normal $\quad S_3/\langle \rho \rangle = \{\langle\rho\rangle, \phi\langle\rho\rangle\} \simeq \mathbb{Z}_2$

S_3 normal $\quad S_3/S_3 = 1.$

2. $H = \{(), (12)(34), (13)(24), (14)(23)\} \triangleleft S_4$. Let $f = (1,2)$ and $r = (1,2,3)$.

Claim: $S_3 \to S_4/H$ determines an isomorphism of groups.

$f \mapsto fH$

$r \mapsto rH$

I've omitted the multiplication table. It's best to use H, $(12)H$, $(13)H$, $(23)H$, $(123)H$, $(132)H$ as representatives.

The mapping is a homomorphism since it's the composition of the natural injection S_3 and the quotient mapping $S_4 \to S_4/H$.

The elements of S_3 have the form $r^i f^j$ with $i \in \{0,1,2\}$ and $j \in \{0,1\}$.
The mapping is an injection since $r^i f^j \in H \iff r^i j^j = 1$. One can see this since all elements of H except the identity move the number 4 while no $r^i f^j$ moves 4. Finally, the map is surjection since $|S_3| = |S_4/H| = 6$. $\square$

3. $H = \langle (234), (34) \rangle < S_4$.

(a) H is not normal in S_4.

Solution: $H = \{(), (234), (243), (34), (23), (24)\}$, and for instance,

$$(12)(234)(12)^{-1} = (12)(234)(12) = (134) \notin H.$$

(b) Find the distinct right cosets of H. Solution:

$H = \{(), (234), (243), (34), (23), (24)\}$

$H(12) = \{(12), (1342), (1432), (12)(34), (132), (142)\}$

$H(13) = \{(13), (1423), (1243), (143), (123), (13)(24)\}$

$H(14) = \{(14), (1234), (1324), (134), (14)(23), (124)\}$

4. Calculate the Smith normal form for L, finding matrices U, V s.t. $ULV = D$

with $D = \operatorname{diag}(d_1, \dots, d_k)$, $d_i \mid d_{i+1}\ \forall i$.

$$L \quad \begin{matrix} 0 & -1 & -1 \\ 4 & -1 & -1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{matrix} \rightarrow \begin{matrix} 4 & -1 & -1 \\ 0 & -1 & -1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{matrix} \rightarrow \begin{matrix} 0 & 0 & -1 \\ -4 & 0 & -1 \\ -2 & 2 & 0 \\ 6 & -2 & 2 \end{matrix} \xrightarrow{\substack{\text{permute columns} \\ \text{negate 1st column}}} \begin{matrix} 1 & 0 & 0 \\ 1 & -4 & 0 \\ 0 & -2 & 3 \\ -2 & 6 & -2 \end{matrix} \xrightarrow{\text{clear first column}} \begin{matrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & -2 & 2 \\ 0 & 6 & -2 \end{matrix} \xrightarrow{\text{clear second column w/ 3rd row}} \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -2 & 2 \\ 0 & 0 & 4 \end{matrix} \xrightarrow{\text{permute rows, negate}}$$

$$U \quad \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} \rightarrow \begin{matrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{matrix} \longrightarrow \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{matrix} \rightarrow \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{matrix} \rightarrow$$

$$V \quad \begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & -1 & 1 \end{matrix} \rightarrow \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -1 \end{matrix} \longrightarrow$$

L $\begin{matrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{matrix} \xrightarrow{\text{clear second row}} \begin{matrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{matrix}$

U $\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & -1 & -2 & 0 \end{matrix} \longrightarrow \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}$

$V \longrightarrow \begin{matrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 4 & 3 \end{matrix}$

Check: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 4 & -1 & -1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & -4 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ ✓

(b) Give an explicit isomorphism of $\mathbb{Z}^4/A$ and $\prod \mathbb{Z}/d_i\mathbb{Z}$.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^3 \arrow[r, "L"] \arrow[d, "V^{-1}"'] & \mathbb{Z}^4 \arrow[r] \arrow[d, "U"] & \mathbb{Z}^4/A \arrow[r] \arrow[d, "??"] & 0 \\
0 \arrow[r] & \mathbb{Z}^3 \arrow[r, "D"] & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}/1\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

This isomorphism is essential given by the matrix U.

$$\mathbb{Z}^4/A \longrightarrow \mathbb{Z}/1\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}$$

$$(a,b,c,d) \longmapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = (b, -c, 2b+3c+d, a+b+c+d) \longmapsto (-c, 2b+3c+d, a+b+c+d)$$

$-c = c \mod 2$