Math 332 HW Solutions

20.8 Apply the Chinese remainder theorem.

$\mathbb{Z}_{200} \cong \mathbb{Z}_8 \times \mathbb{Z}_{25}$ since $200 = 8 \cdot 25$ and $\gcd(8,25)=1$

$\mathbb{Z}_{200} \not\cong \mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ because 8, 5, 5 are not pairwise relatively prime.

20.10 $\mathbb{Z}_4$

n	0	1	2	3
order(n)	1	4	2	4

U_7

n	1	2	3	4	5	6
order(n)	1	3	6	3	6	2

The order of $(a,b) \in \mathbb{Z}_4 \times U_7$ is $\operatorname{lcm}(|a|,|b|)$:

$\mathbb{Z}_4 \backslash U_7$	1	2	3	4	5	6
0	1	3	6	3	6	2
1	4	12	12	12	12	4
2	2	6	6	6	6	2
3	4	12	12	12	12	4

20.11 4 nonisomorphic groups of order 36:

$$\mathbb{Z}_{36}, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9, \quad \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \quad D_{18}$$

These are nonisomorphic by the chinese remainder theorem

This one is not abelian.

1. A, B abelian iff $A \oplus B$ abelian

Pf/ ($\Longrightarrow$) Suppose A, B abelian and let $(a,b), (a',b') \in A \oplus B$.

Then $(a,b)(a',b') = (aa', bb') = (a'a, b'b) = (a',b')(a,b)$.

since A, B abelian

($\Longleftarrow$) Suppose $A \oplus B$ abelian. Let $a, a' \in A$. Then

$$(a, 1_B)\cdot(a', 1_B) = (a', 1_B)\cdot(a, 1_B) \Rightarrow (aa', 1_B) = (a'a, 1_B) \Rightarrow aa' = a'a.$$

Thus, A is abelian. The argument for B is similar. $\square$

2. The total number of necklaces with 6 beads using 2 of each of 3 colors, not counting symmetry is

$$\binom{6}{2}\binom{4}{2}\binom{2}{2} = 15\cdot 6\cdot 1 = 90$$

choose 2 of the 6 spots for blue beads

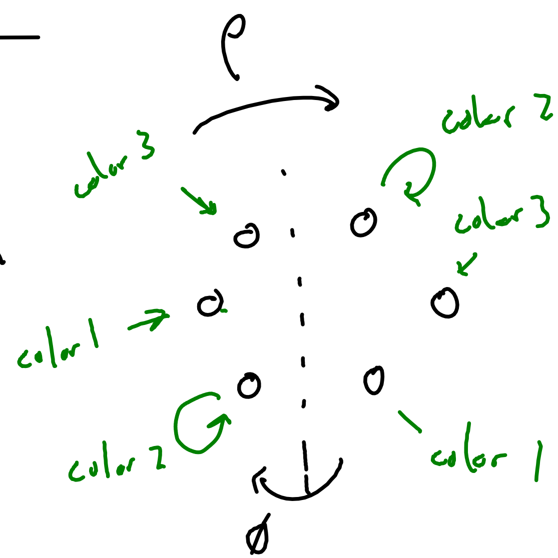
Then choose 2 of the remaining spots for red beads.

Then there are only 2 spots left over for the remaining 2 beads.

(a) Let D_6 act on the necklaces as in class and tally the necklaces fixed by each group element

g	1	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ϕ	★ $\rho\phi$	$\rho^2\phi$	$\rho^3\phi$	$\rho^4\phi$	$\rho^5\phi$
$\mathrm{Fix}\lvert g\rvert$	90	0	0	6	0	0	6	6	6	6	6	6

★ For instance a configuration fixed by $\rho\phi$ has the form



There are 3 choices for color 1, then 2 choices for color 2, then 1 choice for color 3, giving 6 total.

The number of orbits = # of colorings up to D_6 symmetry

$$= \frac{1}{|D_6|} \sum_{g \in D_6} |\mathrm{Fix}(g)| = \frac{1}{12}(90 + 7 \cdot 6) = \frac{132}{12} = 11.$$

(b) Dropping the restriction of using exactly 2 beads of each color, we have 3^6 possible colorings without modding out by symmetry.

g	1	ρ	ρ^2	ρ^3	ρ^4	ρ^5
$\mathrm{Fix}\ g$	3^6	3	3^2	3^3	3^2	3

colorings up to symmetry =

$$\frac{1}{|C_6|} \sum_{g \in C_6} |\mathrm{Fix}(g)| = \frac{1}{6}(3^6 + 2 \cdot 3 + 2 \cdot 3^2 + 3^3)$$

$$= 130$$

(If you kept the 2-bead restriction of part (a), you'd get 15 here.)

3. 4 nonisomorphic groups of order 60

$$\mathbb{Z}_{60}, \quad \mathbb{Z}_2 \times \mathbb{Z}_{30}, \quad D_{30}, \quad A_5$$

$60 = 2 \cdot 30$, but $\gcd(2,30) \neq 1$, so $\mathbb{Z}_{60} \not\cong \mathbb{Z}_2 \times \mathbb{Z}_{30}$

D_{30} and A_5 are non abelian, so they aren't isomorphic to $\mathbb{Z}_{60}$ or $\mathbb{Z}_2 \times \mathbb{Z}_{15}$.

D_{20} and A_5 aren't isomorphic since, for instance, D_{30} has an element of order 30 (a rotation), the maximal order of an element in A_5 is 5.

(An element in A_5 is either the identity, a 3-cycle, a 5-cycle, or a product of two disjoint 2-cycles. These have orders 1, 3, 5, and 2, respectively.)

4. $\phi^{-1}: \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_{420}$

$$\underset{\underset{105}{\shortparallel}}{3\cdot 5\cdot 7},\quad \underset{\underset{140}{\shortparallel}}{4\cdot 5\cdot 7},\quad \underset{\underset{84}{\shortparallel}}{4\cdot 3\cdot 7},\quad \underset{\underset{60}{\shortparallel}}{4\cdot 3\cdot 5}$$

$$-105\cdot 105 + 79\cdot 140 = 35 \qquad 8\cdot 84 - 11\cdot 60 = 12$$

$$11\cdot 35 - 32\cdot 12 = 1 \Rightarrow 11(-105\cdot 105 + 79\cdot 140) - 32(8\cdot 84 - 11\cdot 60) = 1$$

$$\Rightarrow (11)(-105)\cdot 105 + (11\cdot 79)\,140 + (-32\cdot 8)\cdot 84 + (-32\cdot(-11))\cdot 60 = 1$$

$$\Rightarrow \underbrace{-1155\cdot 105}_{-121275} + \underbrace{869\cdot 140}_{121660} + \underbrace{(-256)\cdot 84}_{-21504} + \underbrace{(352)\cdot 60}_{21120} = 1$$

So $\phi^{-1}(a,b,c,d) = -121275a + 121660b - 21504c + 21120d$.

But $\underbrace{85\cdot 105}_{8925} + \underbrace{(-85)\cdot 140}_{-11900} + \underbrace{34\cdot 84}_{2856} + \underbrace{2\cdot 60}_{120} = 1$, too. So we could define

$$\phi^{-1}(a,b,c,d) = 8925\cdot a - 11900b + 2856c + 120d.$$

$$\underbrace{-3 \cdot 105}_{-315} + \underbrace{2 \cdot 140}_{280} + \underbrace{(-1) \cdot 84}_{-84} + \underbrace{2 \cdot 60}_{120} = 1, \text{ too!}$$ (Thanks, Raff.)

So $\phi^{-1}(a,b,c,d) = -315a + 280b - 84c + 120d$ works.

Also: $1 \cdot 105 + (-1) \cdot 140 + (-1) \cdot 84 + 2 \cdot 60 = 1.$ (Thanks, Sei.)

So $\phi^{-1}(a,b,c,d) = 105a - 140b - 84c + 120d$ works

Interesting algorithmic problem: If $\gcd(n_1, \dots, n_k) = 1$, find a "small" $\mathbb{Z}$-linear combination of the n_i giving 1.

5. G a finite abelian group with odd order. Then the product of the elements of G is the identity.

Pf/ Since G has odd order, it has no elements of order 2. So in the product $\prod_{g \in G} g$, each nonidentity element $g \in G$ pairs up with its inverse g^{-1} and cancels. $\square$