Math 332 HW 2

7.7. The center of a group G is an abelian subgroup.

Pf/ Let C be the center of G. By definition of th[e]
identity $e \in G$, it commutes with all elements of G. Hence,
Let $a, b \in C$, and let $x \in G$. Then a, b commute with al[l]
of G, including each other. So $(ab)x = a(bx) = a(xb) = (a$
$= x(ab)$, and $ab = ba$. Also multiplying through by a^{-1} on
and right, $ax = xa \Rightarrow xa^{-1} = a^{-1}x$. Thus, $a^{-1} \in C$. $\square$

9.4 Find the orders of all elements in U_5 and U_8.

Solution/ $U_5 = \{1, 2, 3, 4\}$ and an easy check shows orde[r]
order $(3) = 4$, order $(4) = 2$, and order $(1) = 1$.

$U_8 = \{1, 3, 5, 7\}$: order (3) = order (5) = order $(7) = 2$, and order

9.6 G a group. Prove or give a counterexample:

(a) The subset of all elements of order 2 is a subgroup

False: The identity is required, and it has order 1.

(b) The subset of all elements of finite order is a subgro

False: Consider the free group on letters a and b

the relations $a^2 = b^2 = 1$. Then a, b have finite order

ab does not.

(c) The subset of all elements of order 2 is finite.

False: Let G be the free group on letters a_1, a_2

with relations $a_1^2 = a_2^2 = \cdots = 1$. Or consider $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$

9.8 No group can have exactly 2 elements of order 2.

Pf/ Let G be a group with 2 distinct elements x, y

Two cases: I. x and y commute. Then $(xy)^2 = xyxy = x^2y^2 = 1$

$xy \neq x$ (since $y \neq e$) and $xy \neq y$ (since $y \neq e$).

II. x and y do not commute. Then $(xyx)^2 = xyxxyx = 1$.

$xyx = x \Rightarrow xy = 1 = x^2 \Rightarrow y = x$, a contradiction, and

$xyx = y \Rightarrow yx = x^{-1}y = xy \Rightarrow x, y$ commute, again a contradi

So in either case, we've found a third element of G with ord

1. Find an example of a group G, two elements $a, b \in G$, and

$n \geq 2$ such that $(ab)^n = a^n b^n$ but $ab \neq ba$.

Solution/ In S_4, let $a = (124)$, $b = (234)$. Then $ab = ($

$a^3 b^3 = 1 = (ab)^3$, but $ba = (134) \neq ab$. $\square$

2. Suppose G is a group and $a^2=1$ $\forall a \in G$. Show that G is abelian.

Pf/ Let $a, b \in G$. Then $a^2 = b^2 = 1 \Rightarrow a = a^{-1}, b = b^{-1}$ a

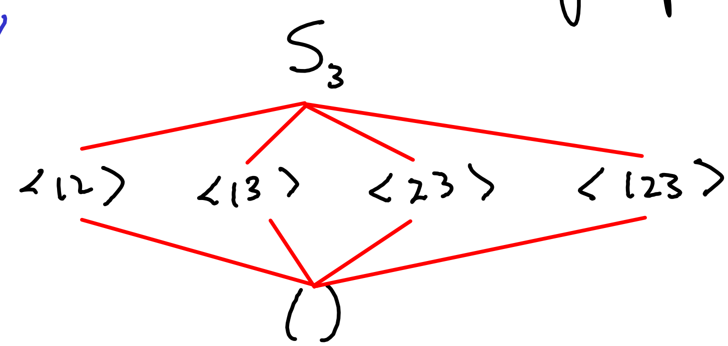
$1 = (ab)^2 = abab \Rightarrow a^{-1} = bab \Rightarrow b^{-1}a^{-1} = ab \Rightarrow ba = ab$.

3. Compute the centers of S_3 and D_4.

Solution/ Both are trivial (by staring at the multiplicat

4. Compute and draw the subgroup lattice for S_3.

Solution/



5. Find all elements of $GL(2,2)$.

Solution/ Need $\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc = 1$ for $a,b,c,d \in \mathbb{Z}_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

$\square$

6. See next page.

6.

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sage: [SL(2,p).order() for p in primes(2,20)]
[6, 24, 120, 336, 1320, 2184, 4896, 6840]
sage: [GL(2,p).order() for p in primes(2,20)]
[6, 48, 480, 2016, 13200, 26208, 78336, 123120]
sage:
sage: [GL(2,p).order()/SL(2,p).order() for p in primes(2,20)]
[1, 2, 4, 6, 10, 12, 16, 18]
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It looks like $|SL(2,p)| = (p-1)p(p+1)$ and $|GL(2,p)| = (p-1)^2 p(p+1)$.