Math 332 HW 12 Solutions

1. Order the terms and find the initial term.

a) $f = x + 3y - x^2 + z^2 - y^3$

lex $\quad -x^2 + x - y^3 + 3y + z^2 \qquad \text{in}(f) = -x^2$

deglex $\quad -y^3 - x^2 + z^2 + x + 3y \qquad \text{in}(f) = -y^3$

degrevlex $\quad -y^3 - x^2 + z^2 + x + 3y \qquad \text{in}(f) = -y^3$

b) $g = x^2yz + xy^6 + 2xy^3 - 4x^2y^3z^2$

lex $\quad -4x^2y^3z^2 + x^2yz + xy^6 + 2xy^3 \qquad \text{in}(g) = -4x^2y^3z^2$

deglex $\quad -4x^2y^3z^2 + xy^6 + x^2yz + 2xy^3 \qquad \text{in}(g) = -4x^2y^3z^2$

degrevlex $\quad xy^6 - 4x^2y^3z^2 + 2xy^3 + x^2yz \qquad \text{in}(g) = xy^6$

2. Simple example of $I = (f_1, \ldots, f_s)$ s.t. $\text{in}\, I \neq (\text{in} f_1, \ldots, \text{in} f_s)$.

Example $\quad I = (x^2, x^2 + y)$. Then $\text{in}\, I = (x^2, y) \neq (\text{in}\, x^2, \text{in}\, x^2 + y) = (x^2)$.

Here, I've taken $>$ to be deglex.

3. $R = k[x,y]$, $I = (y-x^2)$, deglex ordering

a) Use Macaulay's thm. to find simultaneous bases for R/I and $R/\text{in } I$.

Solution: The polynomial $y-x^2$, itself, is a Gröbner basis with initial term $-x^2$. So $\text{in}(I) = (x^2)$. Therefore, a basis for R/I and $R/\text{in } I$ is $\{1, x, y, xy, y^2, xy^2, y^3, xy^3, y^4, \dots\}$, all monomials not divisible by x^2.

b) R/I and $R/\text{in } I$ are not isomorphic as rings.

Pf/ $R/I = k[x,y]/(y-x^2) \cong k[y]$, a domain.

$R/\text{in } I = k[x,y]/(x^2)$ which is not a domain since $x \neq 0$ but $x^2 = 0 \in R/\text{in } I$. $\square$

4. $J \subseteq I$ as ideals in $k[x_1, \dots, x_n]$. Show $\text{in } J = \text{in } I$ iff $I = J$.

Pf/ Suppose $\text{in } J = \text{in } I$ and $J \neq I$. Take $f \in I \setminus J$ of smallest degree. Now $\text{in } J = \text{in } I \Rightarrow \text{in}(f) \in \text{in } J \Rightarrow \exists\, g \in J$ s.t. $\text{in}(g) = \text{in}(f)$.

But then $f-g \notin J$ and $\text{in}(f-g) < \text{in}(f)$. $\divideontimes$. The converse is trivial. $\square$

5. Compute a Gröbner basis for $I = (xy - z^2, y^2 - xz)$ w.r.t. deglex + find $\mathrm{in}\, I$.

Solution/ Start with $G = \{\underline{xy} - z^2, y^2 - \underline{xz}\}$. Underline = leading term.

Let $S = z(xy - z^2) + y(y^2 - xz) = \underline{y^3} - z^3$. Which can't be reduced.

So now $G = (\underline{xy} - z^2, y^2 - \underline{xz}, \underline{y^3} - z^3)$.

Next: $y^2(xy - z^2) - x(y^3 - z^3) = \underline{xz^3} - y^2z^2 \xrightarrow{y^2 - xz} \underline{xz^3} - y^2z^2 + z^2(y^2 - \underline{xz}) = 0$

Next: $y^3(y^2 - \underline{xz}) + xz(\underline{y^3} - z^3) = y^5 - \underline{xz^4} \xrightarrow{y^2 - xz} y^5 - \underline{xz^4} - z^3(y^2 - \underline{xz})$

$= \underline{y^5} - y^2z^3 \xrightarrow{y^3 - z^3} \underline{y^5} - y^2z^3 - y^2(\underline{y^3} - z^3) = 0.$

Therefore, the Gröbner basis for I is $\{\underline{xy} - z^2, y^2 - \underline{xz}, \underline{y^3} - z^3\}$

and $\mathrm{in}\, I = (xy, xz, y^3)$. $\square$

6. Solve the given system of equations by using a computer to find a Gröbner basis w.r.t. lex.

Solution / Using Sage, I computed the Gröbner basis to be

$\{x - z^5, \ y - z^6, \ z^7 - 1\}$. So the solutions are

$$\{(z^5, z^6, z) : z = e^{2\pi i/7} \text{ for } i = 0, 1, \dots, 6\}.$$

7. (a) The solution set is $\{(c_1, \dots, c_n) : c_i \in \{0,1,2\} \text{ for all } i\}$.

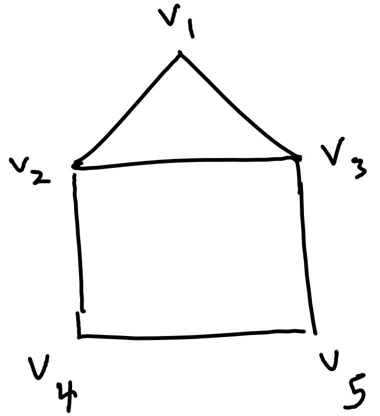
Hence, we get all possible colorings.

(b) Consider the values of $x_i^2 + x_i x_j + x_i^2 - 1$ as x_i, x_j vary over $\mathbb{Z}/3\mathbb{Z}$:

(x_i, x_j)	$x_i^2 + x_i x_j + x_i^2 - 1$
$(0,0)$	-1
$(1,0)$ + $(0,1)$	0
$(1,1)$	-1
$(2,0)$ + $(0,2)$	0
$(1,2)$ + $(2,1)$	0
$(2,2)$	-1

Thus, $x_i^2 + x_i x_j + x_j^2 - 1 = 0$ iff $x_i \neq x_j$.

(c)



Equations:

① $x_1^2 + x_1x_2 + x_2^2 - 1 = 0$

② $x_1^2 + x_1x_3 + x_3^2 - 1 = 0$

③ $x_2^2 + x_2x_3 + x_3^2 - 1 = 0$

④ $x_2^2 + x_2x_4 + x_4^2 - 1 = 0$

⑤ $x_3^2 + x_3x_5 + x_5^2 - 1 = 0$

⑥ $x_4^2 + x_4x_5 + x_5^2 - 1 = 0$

Set $x_1 = 0$, $x_2 = 1$ to get the ideal

$$I = (\underbrace{x_3^2 - 1}_{②},\ \underbrace{x_3 + x_3^2}_{③},\ \underbrace{x_4 + x_4^2}_{④},\ \underbrace{x_3^2 + x_3x_5 + x_5^2 - 1}_{⑤},\ \underbrace{x_4^2 + x_4x_5 + x_5^2 - 1}_{⑥})$$

Computing a Gröbner basis over $\mathbb{Z}/3\mathbb{Z}$ with lex ordering, $x_3 > x_4 > x_5$ gives

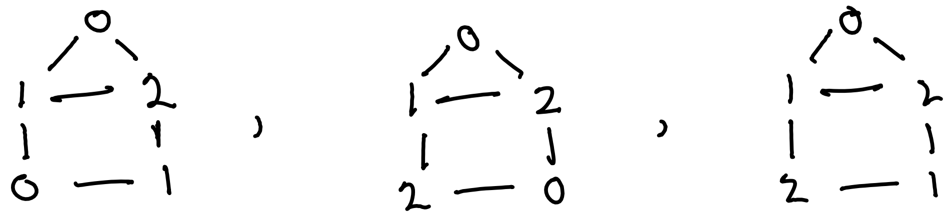
$$x_3 + 1,\ x_4^2 + x_4,\ x_4x_5 - x_4 + x_5 - 1,\ x_5^2 - x_5$$

So $x_3 = -1 = 2 \mod 3$, $x_4(x_4+1) = 0$, $x_5(x_5 - 1) = 0$, $x_4x_5 - x_4 + x_5 + 1 = 0$

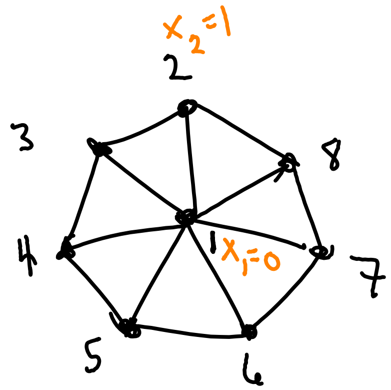
$\Rightarrow$ $x_3 = 2$, $x_4 = 0$ or 2, $x_5 = 0$ or 1, $x_4x_5 - x_4 + x_5 - 1 = 0$

If $x_4 = 0$, then $x_4x_5 - x_4 + x_5 - 1 = 0 \Rightarrow x_5 = 1$. If $x_4 = 2$, then, $x_5 = 0$ or 1

So there are 3 coloring (modulo permutations of the colors:



(d)



Set $x_1=0$, $x_2=1$. Then

$$I=(x_3^2-1,\ x_4^2-1,\ x_5^2-1,\ x_6^2-1,\ x_7^2-1,\ x_8^2-1,\ x_3+x_3^2,\ x_8+x_8^2,$$
$$x_3^2+x_3x_4+x_4^2-1,\ x_4^2+x_4x_5+x_5^2-1,\ x_5^2+x_5x_6+x_6^2-1,$$
$$x_6^2+x_6x_7+x_7^2-1,\ x_7^2+x_7x_8+x_8^2-1)$$

$$x_1^2+x_1x_i+x_i^2=1$$

becomes $x_i^2=1$ if $x_1=0$.

$$x_2^2+x_2x_i+x_i^2=1$$

becomes $x_i+x_i^2=0$ if $x_2=1$

Sage computes the Gröbner basis to be 1

8. I got 8 elements in the Gröbner basis for lex ordering and 3 for degrevlex.