HW 12, due Tuesday, April 28 *Math 332*

1. Order the terms in the following polynomials using lex, deglex, and revlex ordering, in turn. Assume $x > y > z$. What is the initial term in each case? (You can check your work with a computer but only after doing the problem by hand.)

   (a) $f = x + 3y - x^2 + z^2 - y^3$.

   (b) $g = x^2yz + xy^6 + 2xy^3 - 4x^2y^3z^2$.

2. Give a simple example of an ideal $I = (f_1, \dots, f_s)$ and an ordering $>$ such that $\mathrm{in}_>(I) \neq (\mathrm{in}_>(f_1), \dots, \mathrm{in}_>(f_s))$.

3. If $I$ is an ideal in $R = k[x_1, \dots, x_n]$ with term-ordering $>$, Macaulay's theorem says that $R/I$ and $R/\mathrm{in}_>(I)$ are isomophic as $k$-vector spaces but not necessarily as rings. Now let $R = k[x, y]$ with deglex monomial ordering, and let $I = (y - x^2)$.

   (a) Use Macaulay's theorem to exhibit $k$-bases of $R/I$ and of $R/\mathrm{in}_>(I)$ consisting of the same set of monomials.

   (b) Show that $R/I$ and $R/\mathrm{in}_>(I)$ are not isomorphic as rings.

4. Let $J \subseteq I$ be ideals of $k[x_1, \dots, x_n]$. Prove that $\mathrm{in}(J) = \mathrm{in}(I)$ if and only if $J = I$.

5. Compute by hand a deglex Gröbner basis for the ideal

$$I = (xy - z^3, y^2 - xz).$$

   What is $\mathrm{in}(I)$?

6. Solve the following system of equations by using a computer to find a Gröbner basis with respect to lex ordering (an elimination ordering), then back-substituting:

$$x^3 - yz^2 = y^2 - x = z^3 - x^2 = 1 - yz = 0.$$

   What are all the solutions over the complex numbers? What happens if you change the order of the indeterminates?

7. (Kreuzer and Robbiano) **Graph Colorings.**

   Let $\Gamma = (V, E)$ be a graph with vertices $V$ and edges $E$. Suppose $|V| = n$ and there is at most one edge between any two vertices. You are given 3 colors and asked if there is a way to color the vertices of the graph so that no edge connects two vertices of the same color. Call these colorings "acceptable".

   Identify the colors with the elements of the field $k = \mathbb{Z}/3\mathbb{Z}$. Let $R = k[x_1, \dots, x_n]$, and identify a coloring of the graph with an element of $\mathbb{A}^n_k$ in the natural way.

(a) Show that the zeros of the ideal $(x_1^3 - x_1, \ldots, x_n^3 - x_n)$ are exactly the set of all colorings of $\Gamma$.

(b) Prove that in a coloring, the vertices $i$ and $j$ have different colors if and only if the coloring satisfies the equation $x_i^2 + x_i x_j + x_j^2 - 1$.

(c) Draw a simple graph, make the corresponding "coloring ideal", consisting of the equations from part 7b, and compute a Gröbner basis with respect to lex ordering to find all the acceptable colorings. (In practice it probably make sense to choose two vertices connected by an edge, call these $v_1$ and $v_2$, then set the color of $v_1$ to be 0 and the color of $v_2$ to be 1. This amounts to setting $x_1 = 0$ and $x_2 = 1$ in all of the corresponding equations. One thus computes the colorings modulo permutations of $\{0, 1, 2\}$.) To make a polynomial ring over $\mathbb{Z}_3$ in 3 variables and lex ordering in Sage, enter the following:

```
R.<x,y,z> = PolynomialRing(GF(3),3,order="lex")
```

(d) Consider the graph formed by connecting the center of a regular 7-gon to its vertices. It has 8 vertices and 14 edges. Use Gröbner bases to show that this graph has no acceptable colorings.

8. (Cox, Little, and O'Shea) For a surprise, use a computer to find a Gröbner basis for the ideal

$$I = (x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)$$

first using lex ordering, then using degrevlex. How many elements are in the respective Gröbner bases?