Math 332 HW 10 solutions

1. a) Suppose $1 = \sum_{i=1}^m g_i f_i$ for some $g_i \in R$. Then if $f_1(p) = \cdots = f_m(p) = 0$, it follows that $1 = \sum_{i=1}^m g_i(p) f_i(p) = 0$. Thus, $Z(f_1, \ldots, f_m) = \emptyset$. We don't need to use the fact that k is algebraically closed here.

Conversely, suppose $Z(f_1, \ldots, f_m) = \emptyset$. Then $I(Z(f_1, \ldots, f_m)) = I(\emptyset)$ $\Rightarrow \operatorname{rad}(f_1, \ldots, f_m) = R \Rightarrow 1 \in \operatorname{rad}(f_1, \ldots, f_m) \Rightarrow 1^m = 1 \in (f_1, \ldots, f_m)$ $\Rightarrow \exists\, g_i$ s.t. $1 = \sum g_i f_i$. (Here, k must be algebraically closed. For instance, over $\mathbb{R}$, $Z(x^2+1) = \emptyset$, yet $1 \notin (x^2+1)$.)

b) First we show that $(x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal in $R = k[x_1, \ldots, x_n]$.

Define
$$\phi : k[x_1, \ldots, x_n] \longrightarrow k$$
$$f \longmapsto f(a_1, \ldots, a_n)$$

Then $x_i - a_i \in \ker \phi$ for all i, so there is an induced map

$\tilde{\phi} : \dfrac{k[x_1,\dots,x_n]}{(x_1-a_1,\dots,x-a_n)} \longrightarrow k$. Claim: $\tilde{\phi}$ is an isomorphism.

The inverse is $\psi : k \to k[x_1,\dots,x_n]/(x_1-a_1,\dots,x_n-a_n)$.

$$t \longmapsto t$$

This is because in $k[x_1,\dots,x_n]/(x_1-a_1,\dots,x_n-a_n)$, $f(x_1,\dots,x_n) = f(a_1,\dots,a_n)$ (since $x_i = a_i \ \forall i$ in this ring).

Since $k[x_1,\dots,x_n]/(x_1-a_1,\dots,x_n-a_n)$ is isomorphic to field, $(x_1-a_1,\dots,x_n-a_n)$ is maximal.

Conversely, suppose $m \leq R$ is a maximal ideal. Then m is prime, hence radical. By the Nullstellensatz, $I(Z(m)) = m$. Note that $Z(m) \neq \emptyset$ since then $I(Z(m)) = R \neq m$. So let $a = (a_1,\dots,a_n) \in Z(m)$. Then $\{a\} \subseteq Z(m) \Rightarrow I(\{a\}) \supseteq I(Z(m)) = m$. But $x_i - a_i \in I(\{a\})$ $\forall i \Rightarrow x_i - a_i \in m \ \forall i \Rightarrow (x_1-a_1,\dots,x_n-a_n) \subseteq m$.

However, we've just seen $(x_1-a_1, \ldots, x_n-a_n)$ is maximal. Thus, $(x_1-a_1, \ldots, x_n-a_n) = \mathcal{M}$. $\square$

To see the result does not hold when k is not algebraically closed, note that (x^2+1) is maximal in $\mathbb{R}[x]$ since $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$ and $\mathbb{C}$ is a field. $\square$

$$f(x) \mapsto f(i)$$

c) $Z(I) = \emptyset \Rightarrow \text{rad}(I) = I(\emptyset) = R \Rightarrow 1 \in \text{rad}\, I \Rightarrow 1 \in I \Rightarrow I = R$.

If k is not algebraically closed, the result does not necessarily hold. Consider $I = (x^2+1) \subseteq \mathbb{R}[x]$.

2. $I = (x^2+y-3,\ xy^2+2x,\ y^3)$. Suppose $(x,y) \in \mathbb{A}^2$ with $y^3=0$, $xy^2+2x=0$, and $x^2+y-3=0$. Then $y^3=0 \Rightarrow y=0$. The second equation then gives $2x=0$ and the third gives $x^2=3$.

If $2 \neq 0$ we get $x=0$ and $x=3$, a contradiction if $3 \neq 0$.
So if the characteristic of the field is not 2 or 3, we get
$Z(I) = \emptyset$. Hence, if in addition k is algebraically closed, $1 \in I$.
Note that over $\mathbb{Z}/2\mathbb{Z}$, $(1,0)$ is a solution to all 3 equations.
In this case, $1 \notin I$ (since 1 does not vanish at $(1,0)$).
A similar argument works over $\mathbb{Z}/3\mathbb{Z}$.

3 (a) If $k = \mathbb{Z}/p\mathbb{Z}$ for some prime p, then $x^p - x \in I(\mathbb{A}^1_k)$.

(b) Suppose k is infinite. Then $I(\mathbb{A}^n_k) = 0$.

Pf/ We do this by induction on n. First suppose $n=1$, and let $f \in I(\mathbb{A}^1_k)$.
Since a nonzero polynomial in one variable has only a finite number of zeroes, and
since $\mathbb{A}^1_k$ is infinite, $f=0$. Now suppose $n>1$ and $f \in I(\mathbb{A}^n_k)$. Write
$f = \sum_{i=0}^d f_i(x_2, \dots, x_n) x_1^i$. Let $p \in \mathbb{A}^n_k$. Then $\tilde{f}(x_1, p_2, \dots, p_n) = \sum_{i=0}^d f_i(p_2, \dots, p_n) x_1^i$
is a polynomial in one variable with infinitely many zeroes. Hence, $\tilde{f} = 0$.

It follows that $f_i(p_2, \dots, p_n) = 0$ $\forall i$, $\forall (p_2, \dots, p_n) \in \mathbb{A}_k^{n-1}$. By induction, $f_i = 0$ $\forall i$. Hence, $f = 0$. $\square$

4 (a) $k[u,v,w] \longrightarrow k[x,y]$

$$u \longmapsto y - x^2$$
$$v \longmapsto xy$$
$$w \longmapsto x^3 + 2y^2$$

(b) $k[x,y,z]/(y-x^3, z-xy) \longrightarrow k[t]$

$$x \longmapsto t$$
$$y \longmapsto t^3$$
$$z \longmapsto t^4$$

5 (a) $\mathbb{A}^1 \longrightarrow \mathbb{A}^2$

$$t \longmapsto (t^2-1, t(t^2-1))$$

(b) $\mathbb{A}^3_{x,y,z} \supseteq Z(xy-z) \longrightarrow Z(s^2-w, sw-tu) \subseteq \mathbb{A}^4_{s,t,u,w}$

$$(x,y,z) \longmapsto (xy, yz, xz, z^2)$$

6. $k[x,y,z]/(y-x^3, z-xy) \xrightarrow{\varphi} k[t]$ has inverse $k[t] \xrightarrow{\psi} k[x,y,z]/(y-x^3, z-xy)$

$$x \longmapsto t \qquad\qquad t \mapsto x$$
$$y \longmapsto t^3$$
$$z \longmapsto t^4$$

Pf/ Since these are k-algebra homomorphisms, it suffices to show the generators of fixed by $\varphi\circ\psi$ and $\psi\circ\varphi$. First, $(\varphi\circ\psi)(t) = \varphi(x) = t$. Next, $(\psi\circ\varphi)(x) = \psi(x) = t$; $(\psi\circ\varphi)(y) = \psi(t^3) = x^3 = y$; $(\psi\circ\varphi)(z) = \psi(t^4) = x^4$. Then, since $y = x^3$ and $z = xy$ in $k[x,y,z]/(y-x^3, z-xy)$, we get $z = x^4$. So, $(\psi\circ\varphi)(z) = z$. □

7 (a) Let $X \subseteq \mathbb{A}^n$, and let $Y \supseteq X$ be a closed set. Thus, $Y = Z(J)$ for some ideal J. Then, $Y \supseteq X \Rightarrow I(Y) \subseteq I(X) \Rightarrow Z(I(Y)) \supseteq Z(I(X))$, and since Y is an algebraic set, $Y = Z(I(Y))$. Hence, $Y \supseteq Z(I(X))$. □

(b) Let $U \subseteq \mathbb{A}^n$ be a non-empty Zariski open set. So $U = X^c$ for some algebraic set $X \subsetneq \mathbb{A}^n$. Now use the fact that $\mathbb{A}^n$ is irreducible:

$\mathbb{A}^n = \overline{U} \cup X$ and $X \subsetneq \mathbb{A}^n \Rightarrow \mathbb{A}^n = \overline{U}$. $\square$

(c) The closure of $\{(n,n) \in \mathbb{A}^2 : n \in \mathbb{Z}\}$ is the line $L = Z(y-x)$.
One way to see this is to note that L is isomorphic to $\mathbb{A}^1$:

$$\begin{array}{ccc} \mathbb{A}^1 & \to & L \\ t & \mapsto & (t,t) \\ x & \leftarrow & (x,y) \end{array} \quad \text{or algebraically,} \quad \begin{array}{ccc} k[x,y]/(y-x) & \to & k[t] \\ x & \mapsto & t \\ y & \mapsto & t \end{array}$$

We've already seen that the closed subsets of $\mathbb{A}^1$ are $\emptyset$, $\mathbb{A}^1$, and finite sets of points (since non-zero polynomials in $k[x]$ have a finite number of zeros). The closure of $\{(n,n)\}_{n \in \mathbb{Z}}$ has infinitely many points, hence must be all of L.

(d) The closure of $U = \{(a,b) \in \mathbb{R}^2 : a^2+b^2 < 1\} \subseteq \mathbb{R}^2$ is $\mathbb{R}^2$.
To see this, note that every line through the origin contains infinitely many points of U. Arguing as in the previous problem, we see that $\overline{U}$ contains each line through the origin. Hence, $\overline{U} = \mathbb{R}^2$.