HW 10, due Thursday, April 16 *Math 332*

1. The Nullstellensatz.

   For the following problems, assume that k is algebraically closed.

   (a) Let $f_1, \dots, f_m \in R$. Show that the system of equations $f_1 = \cdots = f_m = 0$ has no solutions iff 1 is an R-linear combination of the f_i:

   $$1 = \sum_{i=1}^{m} g_i f_i$$

   for some polynomials $g_i \in R$. The implication still runs in one direction, even if k is not algebraically closed. Which one?

   (b) (This one might be difficult. Can you show that R/I is a field? How does that help?) Show that an ideal $I \subset R$ is maximal iff $I = (x_1 - a_1, \dots, x_n - a_n)$ for some $(a_1, \dots, a_n) \in \mathbb{A}^n_k$. Show by example that this result does not hold if k is not algebraically closed.

   (c) If I is an ideal of R, not equal to R, show $Z(I) \neq \emptyset$. (This result is called the "weak Nullstellensatz".) Again, show this result does not hold if k is not algebraically closed.

2. Is 1 in the ideal $(x^2 + y - 3, xy^2 + 2x, y^3)$? Does the answer depend on k?

3. (a) Show that it is possible for $I(\mathbb{A}^n_k) \neq (0)$.

   (b) Show that $I(\mathbb{A}^n_k) = (0)$ if k is infinite. (Hint: induction on n and use that fact that a nonzero $f \in k[x]$ has a finite number of roots.)

4. For each of the polynomial mappings $X \to Y$, describe corresponding ring homomorphisms, $A(Y) \to A(X)$, using the notation of problem 5.

   (a)

   $$\begin{aligned} \phi : \mathbb{A}^2 &\to \mathbb{A}^3 \\ (x, y) &\mapsto (y - x^2, xy, x^3 + 2y^2) \end{aligned}$$

   (b) $X = \mathbb{A}^1$ and $Y = Z(y - x^3, z - xy) \subset \mathbb{A}^3$

   $$\begin{aligned} \phi : X &\to Y \\ t &\mapsto (t, t^3, t^4) \end{aligned}$$

5. For each of the ring homomorphisms $A(Y) \to A(X)$, describe the corresponding morphism of algebraic sets, $X \to Y$, using the notation of problem 4.

   (a)

$$\begin{aligned} \sigma : k[x,y] &\to k[t] \\ x &\mapsto t^2 - 1 \\ y &\mapsto t(t^2 - 1) \end{aligned}$$

   (b)

$$\begin{aligned} \sigma : k[s,t,u,w]/(s^2 - w, sw - tu) &\to k[x,y,z]/(xy - z) \\ s &\mapsto xy \\ t &\mapsto yz \\ u &\mapsto xz \\ w &\mapsto z^2 \end{aligned}$$

   The morphism constructed here is a mapping of the saddle surface to a surface in $\mathbb{A}^4$.

6. Show that the mapping in 4b, above, is an isomorphism by showing that the induced mappings of coordinate rings is an isomorphism of rings.

7. Zariski closure.

   In a previous problem set, we discussed the Zariski topology on $\mathbb{A}^n$. The closed sets of the topology are taken to be algebraic sets, i.e., sets of the form $Z(I)$ where I is an ideal of $R = k[x_1, \ldots, x_n]$. Consider $\mathbb{A}^n$ with the Zariski topology.

   (a) Let $X \subseteq \mathbb{A}^n$. Show that $Z(I(X))$ is the *closure* of set X. This means that $Z(I(X))$ smallest closed set containing X. (Show that if Y is a closed set containing X, then $Y \supseteq Z(I(X))$.)

   (b) Suppose k is algebraically closed (or just infinite). Show that the closure of any nonempty (Zariski) open set of $\mathbb{A}^n$ is $\mathbb{A}^n$, i.e., every nonempty open set is dense. (Again, this is quite a difference from the usual topology in the case of $k = \mathbb{R}$ or $\mathbb{C}$.) (Hint: Since the ideal for $\mathbb{A}^n$ is (0), which is prime, we know $\mathbb{A}^n$ is irreducible.)

   (c) What is the closure of the set $\{(n,n) : n \in \mathbb{Z}\}$ in $\mathbb{A}^2_{\mathbb{Q}}$?

   (d) What is the closure (in the Zariski topology) of the set

$$U = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 < 1\}$$

   in $\mathbb{R}^2$?