Math 322 lecture for Friday, Week 12

HAMILTONIAN SYSTEMS

Let $E \subseteq \mathbb{R}^{2n}$ be an open subset, and let $H\colon E \to \mathbb{R}$ be a function in $C^2(E)$, i.e., a function whose second partials exist and are continuous. We will write $H = H(x, y)$ where $x, y \in \mathbb{R}^n$. The system

$$x' = (x'_1, \ldots, x'_n) = H_y := \frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \ldots, \frac{\partial H}{\partial y_n}\right)$$

$$y' = (y'_1, \ldots, y'_n) = -H_x := -\frac{\partial H}{\partial x} = -\left(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}\right),$$

is called a *Hamiltonian system* with n degrees of freedom. The function H is called the *Hamiltonian* or *total energy* of the system.

Theorem 1. (Conservation of energy.) For a Hamiltonian system, the total energy H is constant along trajectories.

Proof. Consider a solution trajectory $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^{2n}$. By the chain rule,

$$\begin{aligned}
\frac{d}{dt}H(\gamma(t)) &= \nabla H \cdot \gamma' \\
&= \frac{\partial H}{\partial x} \cdot x' + \frac{\partial H}{\partial y} \cdot y' \\
&= \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} - \frac{\partial H}{\partial y} \cdot \frac{\partial H}{\partial x} \\
&= 0.
\end{aligned}$$

$\square$

This result means that the solutions lie on level sets for H.

Example. Let $H(x, y) := y\sin(x)$, and consider the Hamiltonian system with one degree of freedom

$$\begin{aligned}
x' &= H_y = \sin(x) \\
y' &= -H_x = -y\cos(x),
\end{aligned}$$

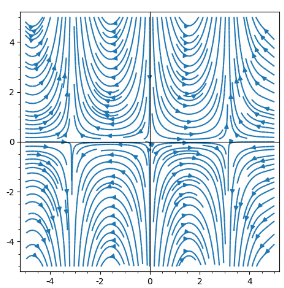
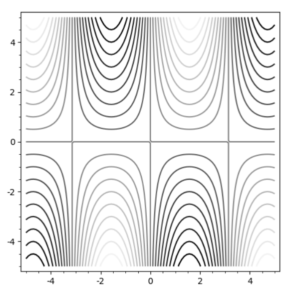
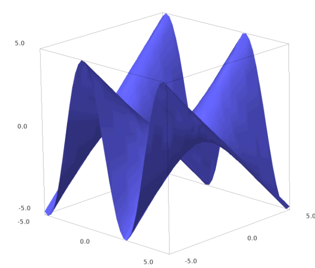
where the x and y subscripts on H denote partial derivatives. We find the critical points:

$$x' = y' = 0 \quad \Rightarrow \quad \sin(x) = y\cos(x) = 0 \quad \Rightarrow \quad x = n\pi \text{ and } y = 0,$$

for $n \in \mathbb{Z}$. Letting $f(x, y) = (\sin(x), -y\cos(x))$, the linearizations at these critical points are

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} = Df(n\pi, 0) \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos(x) & 0 \\ y\sin(x) & -\cos(x) \end{pmatrix} \Bigg|_{(n\pi,0)} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Therefore, the critical points are all topological saddles. Here are pictures of the flow of the system, a contour plot of H (which shows the level sets), and a graph of H:

Critical points. The critical points of a Hamiltonian system $x' = H_x$, $y' = H_y$ occur where where all of the partials of H vanish, i.e., at the critical points for the function H. These are the points where the graph of H,

$$\text{graph}(H) := \left\{(x, y, H(x, y)) \subset \mathbb{R}^{2n+1} : (x, y) \in E\right\},$$

has a "horizontal tangent space", i.e., where the tangent space is given by setting the last coordinate equal to zero. (To parametrize the tangent space, imagine the Jacobian of $(x, y) \to (x, y, H(x, y))$. It is the $2n \times 2n$ identity matrix with an appended row consisting of the partials of H. The columns of this matrix span the tangent space.)

At a critical point p, the geometry of H is determined by its second partials (if these don't also vanish). For the purpose of determining this geometry, by translation, we may assume $p = 0$, the origin, and $H(0) = 0$. Then the second-order Taylor polynomial for H will be

$$Q(x, y) = \frac{1}{2}\frac{\partial^2 H}{\partial x_1^2}(0)x_1^2 + \frac{\partial^2 H}{\partial x_1 \partial x_2}(0)x_1 x_2 + \cdots + \frac{1}{2}\frac{\partial^2 H}{\partial y_n^2}(0)y_n^2.$$

By completing squares and making a linear change of coordinates (or appealing to the spectral theorem for real symmetric matrices), we can transform Q into a function of the form:

$$\widetilde{Q} = v_1^2 + \cdots + v_k^2 - v_{k+1}^2 - \cdots - v_r^2$$

where the new coordinates are $v_1, \ldots, v_r, \ldots, v_{2n}$. The number of pluses and minuses turns out to not depend on the choice of change of coordinates and is the crucial geometric information.

Example. In our earlier example, $H(x, y) = y\sin(x)$, the critical points were found to be $(n\pi, 0)$ for $n \in \mathbb{Z}$. To compute the second-order Taylor polynomial at each of these points, we first compute

$$H_{xx} = -y\sin(x), \quad H_{xy} = \cos(x), \quad H_{yy} = 0.$$

So the second-order approximation of H is

$$\begin{aligned} T(x, y) &= H(n\pi, 0) + \frac{1}{2}H_{xx}(n\pi, 0)(x - n\pi)^2 + H_{xy}(n\pi, 0)(x - n\pi)y + \frac{1}{2}H_{yy}(n\pi, 0)y^2 \\ &= (-1)^n(x - n\pi)y. \end{aligned}$$

Letting

$$u := \frac{1}{2}\left(y + (x - n\pi)\right) \quad \text{and} \quad v := \frac{1}{2}(y - (x - n\pi)),$$

we get

$$u - v = x - n\pi \quad \text{and} \quad u + v = y.$$

Using this change of coordinates, the Taylor polynomial becomes

$$\tilde{T} := (-1)^n(u - v)(u + v) = \pm(u^2 - v^2),$$

and the graph of $\tilde{T}$ is a saddle.

Corollary. Let $p \in \mathbb{R}^{2n}$. Suppose there is a solution $\gamma(t) = (x(t), y(t))$ such that $\gamma(0) \neq p$ but such that $\gamma(t) \to p \in \mathbb{R}^{2n}$ as either $t \to \infty$ or $t \to -\infty$. Then p is not a strict minimum or maximum of H.

Proof. Suppose $\lim_{t\to\infty} \gamma(t) = p$. Using Theorem 1, we know that $H(\gamma(t))$ is constant. Therefore, for all t, we have $H(\gamma(0)) = H(\gamma(t))$. Taking the limit at $t \to \infty$ and using the fact that H is continuous, we get

$$H(\gamma(0)) = \lim_{t\to\infty} H(\gamma(t)) = H(\lim_{t\to\infty} \gamma(t)) = H(p).$$

Thus, in any neighborhood of p, there is a path along which H is constant with value $H(p)$. A similar argument holds in the case $\gamma(t) \to p$ as $t \to -\infty$. $\square$

Theorem 2. Consider a Hamiltonian system with one degree of freedom and total energy function $H(x, y)$. Suppose that H is analytic (i.e., it can be written as a convergent power series at every point in its domain). Then every nondegenerate critical point of the system (points where the linearization has two nonzero eigenvalues) is either a topological saddle or a center. It's a topological saddle if and only if its a saddle for H and it's a center if and only if it's a strict local minimum or maximum for H.

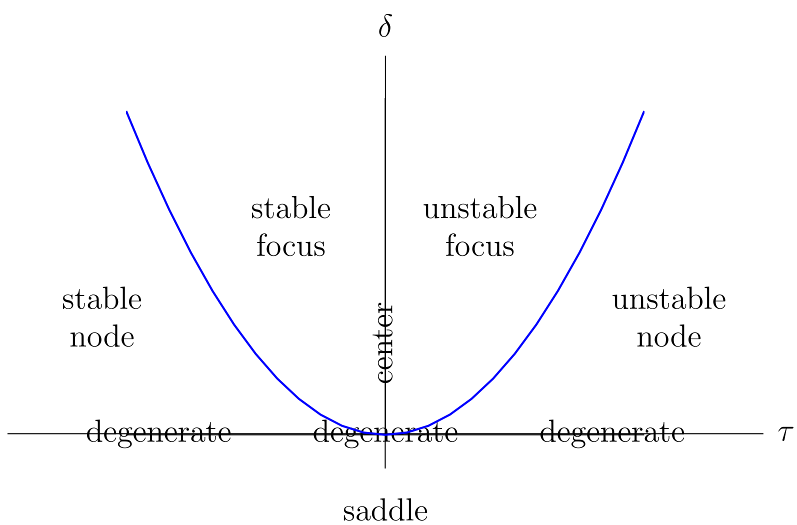
Proof. We classified possible nongenerate critical points earlier in the semester. The above corollary rules out all possibilities except for those listed above. In detail, the linearization of

$$x' = H_y$$
$$y' = -H_x.$$

is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The trace of A, i.e., the sum of its eigenvalues is $\tau = \operatorname{tr}(A) = 0$, and the determinant of A, i.e., the product of its eigenvalues is $\delta = \det(A) = H_{xx}H_{yy} - H_{yx}^2$. Recall our earlier analysis of linear systems in $\mathbb{R}^2$:



In our case, $\tau = 0$, and we see that if $\delta(A) < 0$, the linearized system is a saddle. By Hartman-Grobman, the critical point in the original system is then a (topological) saddle. In the case $\det(A) > 0$, the linearized system is a center. As presented earlier in the course, that means that the critical point in the original system is either a center or a focus. However, the corollary to conservation of energy, proved above, rules out the latter case: by the second derivative test, if $\det(A) > 0$, then H has a strict local maximum or minimum. So the critical point cannot be a focus. $\square$

Newtonian system with one degree of freedom. Consider the equation

$$x'' = f(x)$$

where $f \in C^1(I)$ for some open interval I. We can think of x'' as the acceleration of a particle of mass 1 moving along a line under a force given by f. We can change this into a planar first-order system with the substitution $y = x'$:

$$\begin{aligned} x' &= y \\ y' &= f(x). \end{aligned}$$

To see that this is a Hamiltonian system, we need to find a function $H(x, y)$ such that $H_y = y$ and $H_x = -f(x)$. Integrating the first equation with respect to y gives

$$H(x, y) = \frac{1}{2}y^2 + U(x),$$

for some function U. Taking the partial with respect to x then gives

$$H_x(x, y) = \frac{d}{dx}U(x) = -f(x),$$

and hence,

$$U(x) = -\int_{x_0}^{x} f(s)\, ds.$$

We call

$$T(y) := \frac{1}{2}y^2 = \frac{1}{2}\left(x'\right)^2$$

the *kinetic energy* and $U(x)$ the *potential energy*, and we see the total energy is the sum of the two:

$$H(x, y) = T(y) + U(x).$$

Theorem 3. The critical points of this Newtonian system lie on the x-axis. The point $(x_0, 0)$ is a critical point iff x_0 is a critical point of the function $U(x)$, i.e., iff $U'(x_0) = 0$. Suppose that H is analytic. Then,

1. If x_0 is a strict local maximum for U, then $(x_0, 0)$ is a saddle for the system.
2. If x_0 is a strict local minimum for U, then $(x_0, 0)$ is a center for the system.
3. If x_0 is a horizontal inflection point for U (which means its first nonzero derivative at x_0 of positive order is of an odd order), then $(x_0, 0)$ is a cusp (i.e., two hyperbolic sectors and two separatrices).

Proof. Exercise. □

Example. Consider the case of the undamped pendulum:

$$x'' = -\sin(x).$$

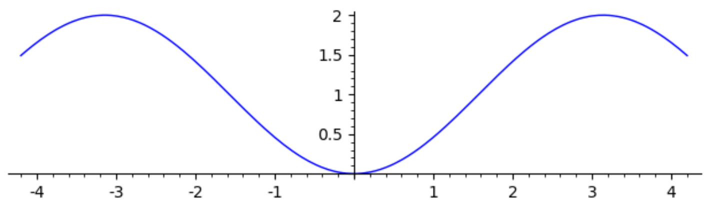
The corresponding first-order planar system is

$$\begin{aligned} x' &= y \\ y' &= -\sin(x). \end{aligned}$$

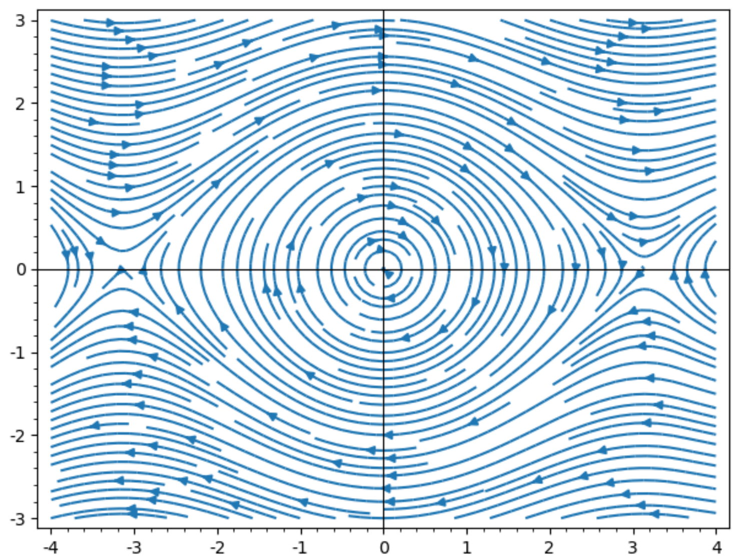
The potential energy function is

$$U(x) = \int_0^x \sin(s)\,ds = 1 - \cos(x).$$

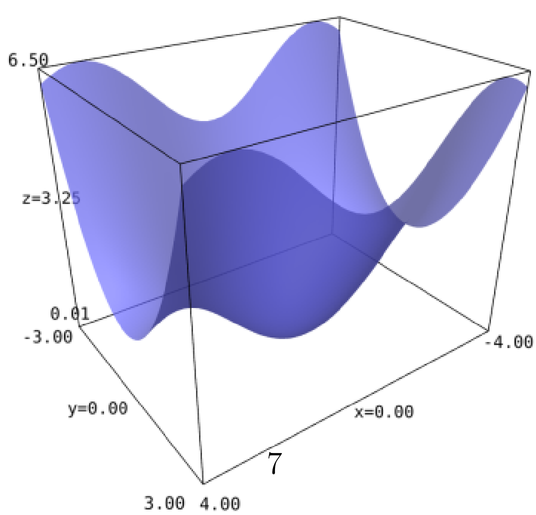
On the next page, we have pictures of both the potential energy and the phase portrait. Try to see how they reflect Theorem 3. Also note the physical meaning of the phase portrait. The x-axis shows the motion of the pendulum. The y-coordinate gives the velocity. The critical points are $n\pi$ for $n \in \mathbb{Z}$ and occur when the pendulum is balanced vertically or hanging straight down. If the velocity is high enough, the pendulum is continuously spinning around in a circle.



Graph of $U(x)$.



Phase portrait.



Graph of H.