Math 322 lecture for Wednesday, Week 11

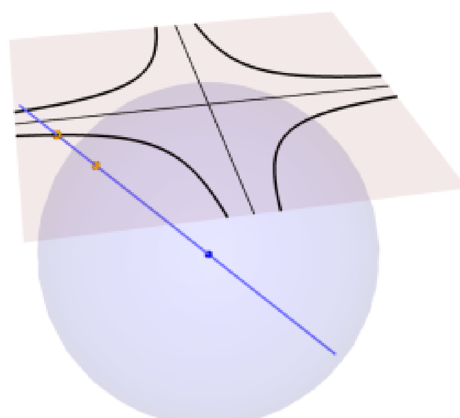
GLOBAL PHASE PORTRAITS

Consider a planar polynomial system:

$$\begin{aligned} x' &= P(x,y) \\ y' &= Q(x,y) \end{aligned} \tag{1}$$

where P and Q are polynomials. Our goal not is to look at critical points of this system "at infinity".

Induced flow on the sphere. Imagine our plane as being the $z = 1$ plane in $\mathbb{R}^3$, which we will denote by Π_z, and then project the flow from the plane to the unit sphere S centered at the origin using a line through the center of the sphere:



This will produce a flow on the sphere that naturally extends to its equator. We think of the points on the equator as points at infinity at our plane, and our goal is to examine the critical points there.

To project $(x, y, 1) \in \Pi_z$ to the sphere, we scale it by $Z \in \mathbb{R}$

$$Z(x, y, 1) = (Zx, Zy, Z) =: (X, Y, Z)$$

to get a point on S. The condition is

$$(Zx)^2 + (Zy)^2 + Z^2 = 1.$$

This means

$$Z = \frac{1}{\sqrt{x^2 + y^2 + 1}}.$$

Therefore, the corresponding point on the sphere is

$$(X, Y, Z) = \frac{1}{\sqrt{x^2 + y^2 + 1}}(x, y, 1).$$

Since

$$x = \frac{X}{Z} \quad \text{and} \quad y = \frac{Y}{Z},$$

we may use (1) to get

$$\begin{aligned} 0 &= QP - PQ \\ &= Qx' - Py' \\ &= Q\left(\frac{X}{Z}\right)' - P\left(\frac{Y}{Z}\right)' \\ &= Q\left(\frac{X'Z - XZ'}{Z^2}\right) - P\left(\frac{Y'Z - YZ'}{Z^2}\right). \end{aligned}$$

Clearing denominators and regrouping, gives

$$QZX' - PZY' + (PY - QX)Z' = 0$$

To think about this geometrically, we'll write this equation as

$$(QZ, -PZ, PY - QX) \cdot (X', Y', Z') = 0.$$

The solution curve $\gamma(t) = (X(t), Y(t), Z(t))$ has velocity vector

$$\gamma'(t) = (X'(t), Y'(t), Z'(t)),$$

and the above equation says that this curve is perpendicular to the vector

$$N = (QZ, -PZ, PY - QX).$$

In preparation for taking a limit as $Z \to 0$, we consider the functions

$$P(x, y) = P\left(\frac{X}{Z}, \frac{Y}{Z}\right) \quad \text{and} \quad Q(x, y) = Q\left(\frac{X}{Z}, \frac{Y}{Z}\right).$$

As functions of X, Y, and Z, these functions now contain powers of Z as denominators. To clear these denominators, let d be the maximum of the degrees of P and Q, and multiply through by Z^d to get new polynomials:

$$P^* := Z^d P, \quad Q^* := Z^d Q, \quad \text{and} \quad N^* := Z^d N = (Q^* Z, -P^* Z, P^* Y - Q^* X).$$

Since we have only scaled N to get N^*, we still have

$$N^* \cdot \gamma'(t) = (Q^* Z, -P^* Z, P^* Y - Q^* X) \cdot \gamma'(t) = 0.$$

What happens as we approach the equator, i.e., as $Z \to 0$? If $P^*Y - Q^*Z \not\to 0$, then $N^* \to (0, 0, a)$ for some nonzero a. In other words, the vector N^* gets closer and closer to pointing straight up. In turn that means that our trajectory gets closer and closer to running parallel to the equator. So at these points, the induced flow on the equator is just a flow along the equator (not across the equator). This says that the place to look for critical points along the equator are the points $(X, Y, 0)$, where

$$P^* Y - Q^* X = 0. \tag{2}$$

Analyzing critical points at ∞. Suppose that using equation (2), we find a point $(a, b, 0)$ of interest. Since the point sits on the sphere, at least one of a and b is nonzero. Say $a \neq 0$. We now use central projection to project our flow onto the plane $x = 1$ in $\mathbb{R}^3$, which we denote by Π_x. Taking a point $(x, y, 1) \in \Pi_z$, we scale it to get

$$\left(1, \frac{y}{x}, \frac{1}{x}\right) \in \Pi_x,$$

which we identify with the point

$$\left(\frac{y}{x}, \frac{1}{x}\right) \in \mathbb{R}^2.$$

In other words, we are identifing Π_x with $\mathbb{R}^2$ using these coordinates. Let

$$u := \frac{y}{x} \quad \text{and} \quad v := \frac{1}{x}.$$

From (1),

$$x' = \left(\frac{1}{v}\right)' = -\frac{v'}{v^2} = P(x, y) = P\left(\frac{1}{v}, \frac{u}{v}\right)$$

$$y' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = Q(x, y) = Q\left(\frac{1}{v}, \frac{u}{v}\right).$$

Projecting our point of interest, $(a, b, 0)$ into the plane $x = 1$ gives the point

$$\left(1, \frac{b}{a}, 0\right).$$

So in the u, v-plane representing Π_x, our job is to analyze the point

$$\left(\frac{b}{a}, 0\right)$$

for the system defined by

$$-\frac{v'}{v^2} = P\left(\frac{1}{v}, \frac{u}{v}\right)$$

$$\frac{u'v - uv'}{v^2} = Q\left(\frac{1}{v}, \frac{u}{v}\right).$$

Example. Consider the saddle

$$x' = -x \tag{3}$$

$$y' = y. \tag{4}$$

So $P(x, y) = -x$ and $Q(x, y) = y$. To find the interesting points on the equator, we consider

$$PY - QX = P\left(\frac{X}{Z}, \frac{Y}{Z}\right)Y - Q\left(\frac{X}{Z}, \frac{Y}{Z}\right)X$$

$$= -\frac{X}{Z}Y - \frac{Y}{Z}X = -2\frac{XY}{Z} = 0.$$

Clearing denominators gives

$$2XY = 0.$$

So either $X = 0$ or $Y = 0$. The corresponding points on the equator are

$$(0, 1, 0) \quad \text{and} \quad (1, 0, 0).$$

Let's look at $(1, 0, 0)$, first. We want to project to the $x = 1$ plane. The mapping of interest is

$$(x, y, 1) \rightsquigarrow \left(1, \frac{y}{x}, \frac{1}{x}\right).$$

Let $u = \frac{y}{x}$ and $v = \frac{1}{x}$ and substitute into our system. The first equation in the system says

$$x' = \left(\frac{1}{v}\right)' = -\frac{v'}{v^2} = -x = -\frac{1}{v}.$$

Therefore,

$$v' = v.$$

Continuing with the second equation in the system:

$$\begin{aligned}
y' &= \left(\frac{u}{v}\right)' \\
&= \frac{u'v - uv'}{v^2} \\
&= \frac{u'v - uv}{v^2} && \text{(since } v' = v\text{)} \\
&= \frac{u' - u}{v} \\
&= y = \frac{u}{v}.
\end{aligned}$$

Therefore, $u' - u = u$, and so

$$u' = 2u.$$

Thus, at the point $(1, 0, 0)$ on the equator, our system looks like the system

$$\begin{aligned}
u' &= 2u \\
v' &= v,
\end{aligned}$$

which is a **source**.

Now let's look at the other interesting point on the equator, $(0, 1, 0)$. The relevant mapping is

$$(x, y, 1) \rightsquigarrow \left(\frac{x}{y}, 1, \frac{1}{y}\right).$$

Now let $u = \frac{x}{y}$ and $v = \frac{1}{y}$ and consider the point $(1, 0)$. Plug these into the system:

$$y' = \left(\frac{1}{v}\right)' = -\frac{v'}{v^2} = y = \frac{1}{v}.$$

Thus,

$$v' = -v.$$

Next,

$$x' = \left(\frac{u}{v}\right)'$$

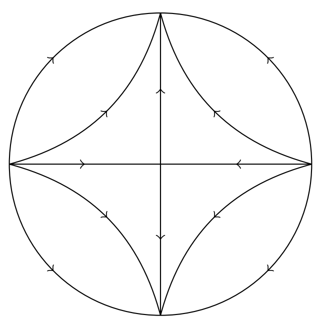
$$
\begin{aligned}
&= \frac{u'v - uv'}{v^2} \\
&= \frac{u'v + uv}{v^2} && \text{(since } v' = -v\text{)} \\
&= \frac{u' + u}{v} \\
&= -x = -\frac{u}{v}.
\end{aligned}
$$

It follows that $u' = -2u$. So the system becomes

$$
\begin{aligned}
u' &= -2u \\
v' &= -v,
\end{aligned}
$$

a **sink**.

Global phase portrait. To get the global phase portrait for a planar system, project the flow onto the upper-hemisphere of the unit sphere, using the process described above, then position yourself way above the north pole, and look down. For the saddle we just considered this looks like:



(If we identify antipodal points on the boundary, we'd get a flow on $\mathbb{P}^2$, the projective plane.)