Math 322 lecture for Monday, Week 10

Equilibrium points for planar systems

Consider a general planar system

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y). \end{aligned}$$

By translating, we can assume that any equilibrium point we are interested in sits at the origin.

Here are some types of equilibrium points.

1. The origin is a **center** if there exists $\delta > 0$ such that every trajectory with initial condition in $B_\delta \setminus \{(0,0)\}$ is a closed curve containing $(0,0)$ in its interior.

2. Let $r(t, r_0, \theta_0)$ and $\theta(t, r_0, \theta_0)$ denote the solution to our system in polar coordinates and with initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$. The origin is a **stable focus** if there exists $\delta > 0$ such that $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$ imply $r(t, r_0, \theta_0) \to (0,0)$ and $|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to \infty$. It is an **unstable focus** if the same holds as $t \to -\infty$.

3. The origin is a **stable node** if there exists $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$, we have $r(t, r_0, \theta_0) \to (0,0)$ as $t \to \infty$ and $\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ exists. In other words, the trajectories approach the origin with a well-defined tangent. It's an **unstable node** if the same holds with $t \to -\infty$. A node is called *proper* if every ray through the origin is tangent to some trajectory.

4. The origin is a **topological saddle** if it is locally homeomorphic to a saddle for a linear system.

5. The origin is a **center-focus** if there exists a sequence of closed solution curves Γ_n with Γ_{n+1} in the interior of Γ_n such that $\Gamma_k \to (0,0)$ as $k \to \infty$ and such that every solution with initial condition between Γ_n and Γ_{n+1} spirals toward either Γ_n or Γ_{n+1} as $t \to \pm\infty$.

Summary of results for hyperbolic equilibria.

Let $x_0 = (0,0)$ be a hyperbolic equilibrium point and assume that P and Q are continuously differentiable. Then

1. The point x_0 is a topological saddle if and only if the linearized system has a saddle at the origin. (This follows from Hartman-Grobman.)

2. If the linearized system has a center, then x_0 is either a center, a focus, or a center-focus. The case of a center-focus cannot occur if P and Q are *analytic* at x_0, i.e., if they can be expressed as power series that converge in some disc about the origin.

3. If x_0 is a node then the linearized system it's a node or a focus for the nonlinear system. Similarly, if it's a focus for the linearized system, then its a node or focus for the nonlinear system. If f has continous second partials, then if x_0 is a node for the linearized system, it is also a node for the nonlinear system, and similarly for foci. (See our text, Example 5, Section 2.10.)

Example. Here is an example of a center-focus:

$$x' = -y + x\sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$y' = x + y\sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right).$$

for $x^2+y^2 \neq 0$, and with $f(0,0) = (0,0)$ where f is the right-hand side of the above. (In particular, it turns out that f is not analytic at the origin.) Changing to polar coordinates gives the system

$$r' = r^2 \sin\left(\frac{1}{r}\right)$$

$$\theta' = 1$$

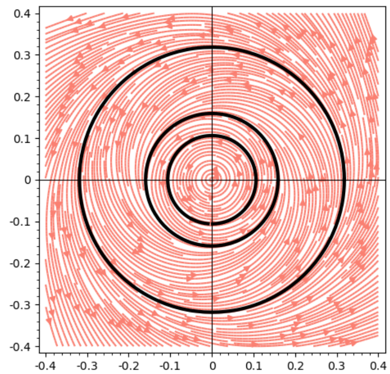
for $r > 0$, and $r' = 0$ for $r = 0$. So $\theta = t + \theta_0$, and if $\sin(1/r) = 0$, i.e., if $r = \frac{1}{n\pi}$ for any $n \in \mathbb{Z}_{>0}$, we have $r' = 0$. So the circles of radius $\frac{1}{n\pi}$ are trajectories. If

$$n\pi < \frac{1}{r} < (n+1)\pi,$$

i.e., if

$$\frac{1}{(n+1)\pi} < r < \frac{1}{n\pi},$$

then $r' < 0$ if n is odd and $r' > 0$ if n is even. Which means the trajectories will either spin inwards or outwards towards one of the circular trajectories. A partial picture appears below:



Nonhyperbolic equilibria. Please set our text, Section 2.11 for a description of possible behaviors for a nonhyperbolic equilibrium point for a two-dimensional system. In particular, please learn the meaning of the following terms: sector, hyperbolic sector, parabolic sector, elliptic sector, saddle-node.

Note the comment on p. 150: if the linearized system is nonzero, the only types of equilibrium points that can occur beside those already mentioned for analytic systems are saddle-nodes, critical points with elliptic domains, and cusps. The book gives examples of each of these:

saddle-node (two hyperbolic sectors, one parabolic sector):

$$\begin{aligned} x' &= x^2 \\ y' &= y \end{aligned}$$

critical point with elliptic domain (one elliptic sector, one hyperbolic sector, two parabolic sectors, four separatrices):

$$\begin{aligned} x' &= y \\ y' &= -x^3 + 4xy \end{aligned}$$

cusp (two hyperbolic sectors, two separatrices):

$$\begin{aligned} x' &= y \\ y' &= x^2 \end{aligned}$$