Math 322 lecture for Monday, Week 9

Liapunov functions and stability

Definition. An equilibrium point x_0 for a system $x' = f(x)$ is *stable* if for each open neighborhood U of x_0, there exists another open neighborhood W of x_0 such that if $p \in W$, then $\phi(t,p) \in U$ for all $t \geq 0$. Otherwise, x_0 is *unstable*. We say x_0 is *asymptotically stable* if it has an open neighborhood W such that

$$\lim_{t\to\infty} \phi_t(p) = x_0$$

for all $p \in W$.

Facts.

(a) Surprisingly, an equilibrium point can be both unstable and asymptotically stable! We'll see an example in the homework.

(b) Suppose x_0 is a hyperbolic equilibrium point, i.e., it's linearized system has no eigenvalues with real part equal to 0. To analyze the stability of x_0, we use Hartman-Grobman to replace the system $x' = f(x)$ with its linearization $x' = Df_{x_0}(x)$ at x_0. If all eigenvalues of Df_{x_0} have negative real part, then x_0 is stable and asymptotically stable, and the approach of a trajectory to x_0 is exponential in time. Otherwise, some eigenvalue has positive real part, and x_0 is unstable.

(c) In any case, it turns out that if an equilibrium point x_0 is stable, then no eigenvalue of Df_{x_0} has positive real part (even in the non-hyperbolic case).

Liapunov functions. Let x_0 be an equilibrium point. Suppose there is a way to assign a smoothly changing "temperature" to each point in E such that: (i) the temperature at x_0 is 0, (ii) the temperature at every other point is positive. Could we determine stability only knowing the temperatures along trajectories? This is the idea behind the notion of a Liapunov function. (Below, we label the temperature function by V.)

Given $V\colon E \to \mathbb{R}$ and $p \in E$, we define

$$\dot{V}(p) = \frac{d}{dt} V(\phi_t(p))\bigg|_{t=0}.$$

Thus $\dot{V}(p)$ tells us how fast the temperature is changing along the solution trajectory as it passes through p.

Theorem. Let $f \in C^1(E)$ and $f(x_0) = 0$. Let $V\colon E \to \mathbb{R}$ also be C^1 (continuously differentiable). Suppose that $V(p) \geq 0$ and $V(p) = 0$ if and only if $p = x_0$. Then:

(a) If $\dot{V}$ is negative semidefinite ($\dot{V}(p) \leq 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is stable.

(b) If $\dot{V}$ is negative definite ($\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$) then x_0 is asymptotically stable.

(c) If $\dot{V}$ is positive definite ($\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$), then x_0 is unstable.

Definition. A function satisfying the hypotheses of the previous theorem is called a *Liapunov function.*

Happily, thanks to the chain rule, the conditions on $\dot{V}$ in the theorem can be verified *without solving the system*:

Proposition. With V as above,

$$\dot{V}(p) = \nabla V(p) \cdot f(p).$$

Proof. Let $\psi(t) := \phi_t(p)$. Apply the chain rule:

$$\begin{aligned}
J(V \circ \psi)(0) &= JV(\psi(0))J\psi(0) \\
&= JV(p)J\psi(0) \\
&= \begin{pmatrix} \frac{\partial V}{\partial x_1}(p) & \cdots & \frac{\partial V}{\partial x_n}(p) \end{pmatrix} \begin{pmatrix} \psi_1'(0) \\ \vdots \\ \psi_n'(0) \end{pmatrix} \\
&= \nabla V(p) \cdot \psi'(0).
\end{aligned}$$

Now, ψ is the solution to the system $x' = f(x)$ with initial condition p. Therefore, $\psi'(t) = f(\psi(t))$, and $\psi'(0) = f(\psi(0)) = f(p)$. The result follows. □

Example. Consider the system

$$\begin{aligned}
x' &= -y^3 \\
y' &= x^3.
\end{aligned}$$

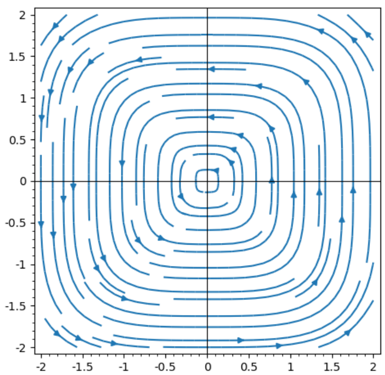
The origin is a non-hyperbolic equilibrium point and

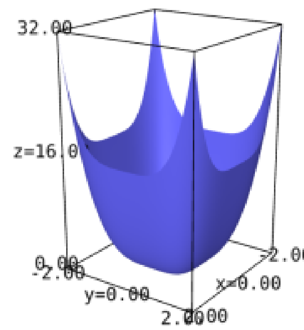
$$V(x, y) = x^4 + y^4$$

is a Liapunov function for that point. (Clearly, V is smooth and $V(x, y) \geq 0$ with equality only at the origin.) For any trajectory $(x, y) = (x(t), y(t))$, we have

$$\dot{V}(x, y) = 4x^3x' + 4y^3y' = 4x^3(-y^3) + 4y^3(x^3) = 0.$$

Hence, the origin is stable. In fact, our calculation shows that $V(\phi_t(p))$ is a constant as a function of t. In other words, trajectories (solutions) sit on level sets for V, as seen in the following:





Proof of theorem. We may assume $x_0 = 0 \in \mathbb{R}^n$ is the equilibrium point.

(1) Suppose that $\dot{V}(p) \leq 0$ for all $p \in E \setminus \{x_0\}$. Choose $\varepsilon > 0$ such that the open ball $B_\varepsilon(x_0)$ of radius ε centered at x_0 is contained in E. Let

$$\overline{B_\varepsilon(x_0)} := \{x \in \mathbb{R}^n : |x - x_0| \leq \varepsilon\} .$$

Replacing ε by $\varepsilon/2$, if necessary, we may assume $\overline{B_\varepsilon(x_0)} \subset E$. Let

$$\alpha := \min_{|x|=\varepsilon} V(x),$$

the minimum of V on the boundary of $\overline{B_\varepsilon(x_0)}$. The function V achieves its minimum on the boundary since V is continuous and the boundary is compact (closed and bounded). Since the minimum is achieved at some point on the boundary and V is strictly greater than 0 away from the origin, we have $\alpha > 0$.

Define

$$W := \{x \in B_\varepsilon(x_0) : V(x) < \alpha\} .$$

We think of W as the set of points in B_ε whose "temperature" is less that α, the minimum temperature on the boundary of B_ε. Then W is an open[1] neighborhood of the origin, and no solution starting at a point in W can leave W since V is nonincreasing on solution curves. Thus x_0 is stable.

[1]The set W is open since $W = V^{-1}((-\infty, \alpha))$, and by definition of continuity, the inverse image of an open subset under a continuous function is continuous.

(2) Suppose now that $\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$. As in the proof for part (1), we choose $\varepsilon > 0$ so that $\overline{B_\varepsilon(x_0)} \subset E$. We let

$$\alpha := \min_{|x|=\varepsilon} V(x),$$

and take

$$W := \{x \in B_\varepsilon(x_0) : V(x) < \alpha\}.$$

Since $\dot{V}(p) < 0$ for all $p \in E \setminus \{x_0\}$, we saw in the proof of part (1) that solution trajectories starting in W never leave W. We would like to show that $\lim_{t\to\infty} \phi_t(p) = 0$ for all $p \in W$. Pick any sequence $t_1 < t_2 < \ldots$ such that $t_n \to \infty$, and consider the sequence

$$\{\phi(t_n, p)\}.$$

By part (1), this sequence never leaves W, and hence it is contained in the closure $\overline{W} \subseteq \overline{B_\varepsilon(x_0)}$, which is compact. So by the Bolzano-Weierstrass theorem, there exists a convergent subsequence. This means that there is a subsequence t_{n_k} such that

$$\lim_{k\to\infty} \phi(t_{n_k}, p) = q$$

for some $q \in \overline{W}$. For ease of writing, replace our original sequence with the subsequence $\{t_{n_k}\}_k$. We then have

$$\lim_{n\to\infty} \phi(t_n, p) = q.$$

We would like to show that $q = x_0 = 0$, and we will do this by contradiction. Suppose that $q \neq 0$. Then $V(q) > 0$. Also since V is strictly decreasing along trajectories, we have

$$V(q) > V(\phi(1, q)).$$

Since $\lim_{n\to\infty} \phi(t_n, p) = q$, by continuity of solutions with respect to both time and initial conditions, and by continuity of V, there exists an integer N large enough so that $\phi(t_N, p)$ is close enough to q so that $V(\phi(1, \phi(t_N, p)))$ is close enough to $V(\phi(1, q))$ so that

$$V(\phi(1 + t_N, p)) = V(\phi(1, \phi(t_N, p)) < V(q).$$

Since $t_n \to \infty$, we can find M such that $t_M > 1 + t_N$. Then, since V is strictly decreasing along trajectories, we have

$$V(q) > V(\phi(1 + t_N, p)) > V(\phi(t_M, p))$$

This is a problem: since V strictly decreases along trajectories and V is continuous, we have that the sequence $\{V(\phi(t_n, p)\}$ is strictly decreasing and converges to $V(q)$. So in contradiction to the inequalities displayed above,

$$V(\phi(t_M, p) > V(q).$$

We have shown that $q = 0$ and that there is a sequence $\{t_n\}$ such that $\lim_{n\to\infty} \phi(t_n, p) = q = 0$. We now need to show $\lim_{t\to\infty} \phi(t, p) = x_0 = 0$. If not, there exists an $\eta > 0$ such that for all n, there exists $s_n > n$ such that

$$|\phi(s_n, p)| \geq \eta > 0. \tag{1}$$

We may assume that the sequence s_n is increasing. However, by Bolzano-Weierstrass, there again exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\phi(s_{n_k}, p)$ converges, and as we have seen, it must converge to 0. But that's impossible in light of (1).

(3) Finally, now suppose that $\dot{V}(p) > 0$ for all $p \in E \setminus \{x_0\}$. Choose $\varepsilon > 0$ such that $\overline{B_\varepsilon(0)} \subset E$. We'll show that given any point $p \in E$, we have that $\phi_t(p)$ leaves $B_\varepsilon(0)$ at some point, i.e., there exists $t \geq 0$ such that $|\phi_t(p)| > \varepsilon$. Hence, x_0 is unstable.

Given $p \in E \setminus \{0\}$, since V is strictly increasing on trajectories,

$$V(\phi_t(p)) > V(\phi_0(p)) = V(p) > 0$$

for all $t > 0$. Thus, $\phi_t(p)$ is bounded away from 0. Say $|\phi_t(p)| \geq \eta > 0$ for all $t \geq 0$. If $\eta \geq \varepsilon$, then we are done since $|p| = |\phi_0(p)| \geq \eta > \varepsilon$, which says p is already out of $B_\varepsilon(x_0)$. Otherwise, define

$$m := \min_{y : \eta \leq |y| \leq \varepsilon} \dot{V}(y),$$

which exists since $\dot{V}$ is continuous and y is restricted to a compact set. In fact, for that same reason, $m = \dot{V}(q)$ for some point in the set over which we are minimizing. Therefore, $m > 0$. Supposing for contradiction that $\phi_t(p)$ stays inside $B_\varepsilon(x_0)$ for all $t \geq 0$, we have $\dot{V}(\phi_t(p)) \geq m$ for all $t \geq 0$. Hence,

$$V(\phi_t(p)) - V(p) = V(\phi_t(p)) - V(\phi_0(p)) = \int_{s=0}^{t} \dot{V}(\phi_s(p))\, ds \geq mt \to \infty$$

as $t \to \infty$. But since V is continuous, it achieves a maximum on $\overline{B_\varepsilon(x_0)}$—a contradiction. □

Example. Consider the system

$$\begin{aligned} x' &= -2y + yz \\ y' &= x - xz \\ z' &= xy. \end{aligned}$$

The Jacobian at the origin is

$$J(0) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det \begin{pmatrix} -x & -2 & 0 \\ 1 & -x & 0 \\ 0 & 0 & -x \end{pmatrix} = -x^3 - 2x = -x(x^2 + 2).$$

So the eigenvalues are $0, \pm\sqrt{2}i$. So the origin is a nonhyperbolic equilibrium point. To determine stability, we look for a suitable Liapunov function. We guess a function of the form

$$V = ax^2 + by^2 + cz^2$$

with positive constants a, b, c. We have

$$\begin{aligned} \dot{V} &= 2axx' + 2byy' + 2czz' \\ &= 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy) \\ &= 2(-2a + b)xy + 2(a - b + c)xyz. \end{aligned}$$

Take $a = c = 1$ and $b = 2$, and we get $V = x^2 + 2y^2 + z^2$ with $\dot{V} = 0$. This means that trajectories stay on the ellipsoids that are level sets of V. □