Math 322 lecture for Monday, Week 8

STABLE MANIFOLD THEOREM

Review of stable, unstable, and center subspaces. Consider the linear system $x' = Ax$ for some $A \in M_n(\mathbb{R})$. Suppose that the generalized eigenvectors and their corresponding eigenvalues for A are $u_j + iv_j$ and $\lambda_j = a_j + ib_j$, respectively, for $j = 1, \dots, n$. Thus, putting these vectors as columns in a matrix P, we have $P^{-1}AP = J$ where J is the Jordan form of A. The generalized eigenvectors $u_j + iv_j$ for which $b_j \neq 0$ come in conjugate pairs since A is a real matrix. Then the **stable, unstable, and center subspaces** for the system are, respectively,

$$\begin{aligned} E^s &:= \operatorname{Span}\{u_j, v_j : a_j < 0\} \\ E^u &:= \operatorname{Span}\{u_j, v_j : a_j > 0\} \\ E^c &:= \operatorname{Span}\{u_j, v_j : a_j = 0\}. \end{aligned}$$

Recall that up to a change of coordinates, the solution to the system is e^{Jt} and that for a Jordan block corresponding to $\lambda_j = a_j + ib_j$, we can factor out $e^{\lambda_j t} = e^{a_j t}(\cos(b_j t) + i \sin(b_j t))$, leaving a matrix that is polynomial in t:

$$e^{J_\ell(\lambda_j)t} = e^{\lambda_j t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{\ell-1}}{(\ell-1)!} \\ 0 & 1 & t & \dots & \dots & \frac{t^{\ell-2}}{(\ell-2)!} \\ 0 & 0 & 1 & \dots & \dots & \frac{t^{\ell-3}}{(\ell-3)!} \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & t \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Thus, it is the signs of the a_j that determine the long-term behavior of the system.

Theorem. (Stable manifold theorem.) Let $E \subseteq \mathbb{R}^n$ and let $f \in C^1(E)$. Suppose that $f(0) = 0$ and that Df_0 has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Let ϕ be the flow for the system $x' = f(x)$. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linearized system $x' = Df_0(x)$ at 0 and there exists an $(n - k)$-dimensional differentiable manifold U tangent to the unstable space E^u of the linearized system. Further

$$\lim_{t \to \infty} \phi_t(p) = 0$$

for any $p \in S$ and

$$\lim_{t \to -\infty} \phi(p) = 0$$

for any $p \in U$.

Remark. To apply this theorem to an arbitrary equilibrium point x_0, make the change of coordinates $x \mapsto x - x_0$, find the stable and unstable manifolds at the origin, and translate back $x \mapsto x + x_0$.

Example. The system

$$\begin{aligned} x' &= -x - y^2 \\ y' &= y + x^2 \end{aligned}$$

has an equilibrium point at the origin. The Jacobian for $f(x, y) = (-x - y^2, y + x^2)$ is

$$Jf(x, y) = \begin{pmatrix} -1 & -2y \\ 2x & 1 \end{pmatrix}.$$

Therefore,

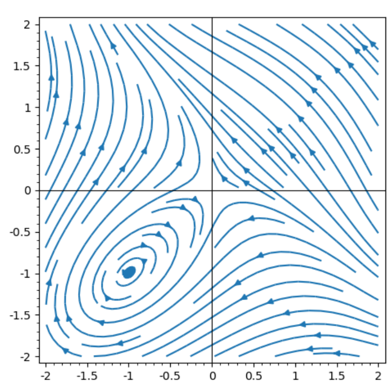
$$Jf(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the linearized system is

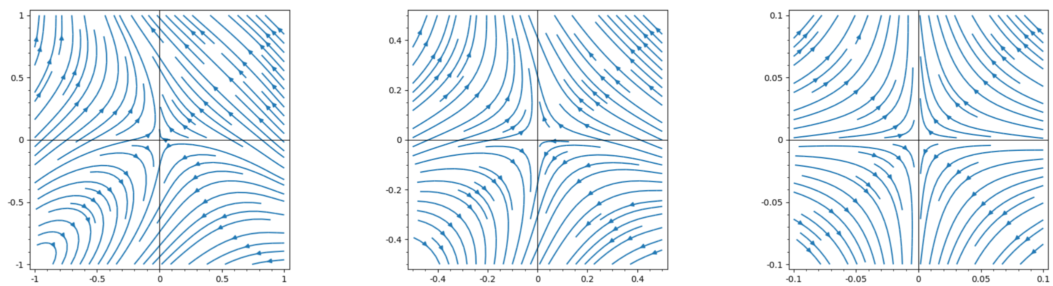
$$\begin{aligned} x' &= -x \\ y' &= y. \end{aligned}$$

(The linearized system in this case is easy to read off of the original system in this case since the equilibrium point is the origin and f has components that are polynomials since f is it's own Taylor expansion at the origin.)

The main thing that concerns us, though, is that the eigenvalues for $Df_{(0,0)}$ are ± 1. The eigenspace for -1 is spanned by $(1, 0)$, i.e., the x-axis, and the eigenspace for 1 is spanned by $(0, 1)$, the y-axis. So a stable manifold for our original system should be tangent to the x-axis and an unstable manifold should be tangent to the y-axis at the origin. Here is a picture of the flow of the vector field f:



From the picture, we can see that the vector field is not tangent to the stable and unstable spaces for the linearized system at the origin everywhere. The stable manifold theorem is a statement about what is happening locally, very close to the equilibrium point. Below, we zoom in on the origin:



Sketch of proof of the stable manifold theorem. The proof of the stable manifold theorem, like the proof of the fundamental existence and uniqueness theorem can be done by the method of successive approximations.

We start with some "pre-processing": As mentioned above, if the equilibrium point x_0 is not the origin, first replace x by $x - x_0$. Suppose that has been done. Second, write

$$x' = f(x) = Jf(0)x + (f(x) - Jf(0)x)\,.$$

Defining $F(x) := f(x) - Jf(0)x$, our system becomes

$$.$$

Third, choose an $n \times n$ real matrix P such that

$$P^{-1}Jf(0)P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A has k eigenvalues with negative real parts and B has $n-k$ eigenvalues with positive real parts. Finally, make the change of variables $y = P^{-1}x$. Then

$$\begin{aligned} x' = Jf(0)x + F(x) \quad &\Rightarrow \quad Py' = Jf(0)Py + F(Py) \\ &\Rightarrow \quad y' = P^{-1}Jf(0)Py + P^{-1}F(Py). \end{aligned}$$

Define $G(y) = P^{-1}F(Py)$ to get the system

$$y' = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} y + G(y). \tag{1}$$

Each step we've made is reversible. So solving this system is equivalent to solving the original system.

We now find stable and unstable manifolds through the method of successive approximations. Define

$$U(t) := \begin{pmatrix} e^{At} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V(t) := \begin{pmatrix} 0 & 0 \\ 0 & e^{Bt} \end{pmatrix}$$

so that

$$e^{\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)t} = U(t) + V(t).$$

For $t \in \mathbb{R}$ and $a \in \mathbb{R}^n$, define an operator T on $\mathbb{R}^n$-valued functions u with domain in a region near the origin in $\mathbb{R} \times \mathbb{R}^n$ by

$$(Tu)(t,a) := U(t)a + \int_{s=0}^{t} U(t-s)G(u(s,a))\,ds - \int_{s=t}^{\infty} V(t-s)G(u(s,a))\,ds. \quad (2)$$

Now use the method of successive approximations starting with

$$u^{(0)}(t,a) = 0 \in \mathbb{R}^n.$$

Calculations like those we did for the proof of the fundamental existence and uniqueness theorem show the approximations $u^{(m)}(t,a)$ converge to a fixed point $u(t)$ of T for t in a small interval about the origin and for a restricted to a sufficiently small neighborhood of the origin in $\mathbb{R}^n$.

A stable manifold for equation (1) is given as the set of points

$$(a_1, \dots, a_k, u_{k+1}(0, a_1, \dots, a_k, 0, \dots, 0), \dots, u_n(0, a_1, \dots, a_k, 0, \dots, 0))$$

as $(a_1, \dots, a_k)$ varies in a neighborhood of the origin in $\mathbb{R}^k$. We get a stable manifold for the original system by applying P to these points, since $y = P^{-1}x$, then translating back $x \mapsto x + x_0$, if the original equilibrium point was not the origin.

To find an unstable manifold, replace t by $-t$ to get the system

$$y' = -\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} y - G(y),$$

since $(y(-t))' = -y'(-t)$. However, now note that $-A$ has k positive eigenvalues and $-B$ has $n-k$ negative eigenvalues, so to apply the above argument, we need to swap coordinates $\phi\colon y \mapsto (y_{k+1}, \dots, y_n, y_1, \dots, y_k)$ to get the system

$$(\phi(y))' = \begin{pmatrix} -B & 0 \\ 0 & -A \end{pmatrix} \phi(y) - G(\phi(y)),$$

apply the method of successive approximations, then swap back by applying ϕ^{-1} to the points in the resulting manifold. $\square$

As evidence for the reasonableness of the method presented in the sketch above suppose that $y(t)$ is a solution to equation (2) with (i) initial condition $y(0)$ close to 0 and such that (ii) $y(t)$ is bounded as $t \to \infty$. We will show that y must satisfy equation (2) (and thus will be a fixed point of the iterative process). Let

$$M := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

so that the system becomes

$$y' = My + G(y).$$

Then

$$\begin{aligned}
y' = My + G(y) \quad &\Rightarrow \quad e^{-Mt}y' = e^{-Mt}My + e^{-Mt}G(y) \\
&\Rightarrow \quad e^{-Mt}y' = Me^{-Mt}y + e^{-Mt}G(y) \\
&\Rightarrow \quad e^{-Mt}y' - Me^{-Mt}y = e^{-Mt}G(y) \\
&\Rightarrow \quad (e^{-Mt}y)' = e^{-Mt}G(y) \\
&\Rightarrow \quad \int_{s=0}^{t} (e^{-Ms}y(s))'\,ds = \int_{s=0}^{t} e^{-Ms}G(y(s)\,ds \\
&\Rightarrow \quad e^{-Mt}y(t) - y(0) = \int_{s=0}^{t} e^{-Ms}G(y(s))\,ds \\
&\Rightarrow \quad y(t) - e^{Mt}y(0) = \int_{s=0}^{t} e^{M(t-s)}G(y(s))\,ds \\
&\Rightarrow \quad y(t) = e^{Mt}y(0) + \int_{s=0}^{t} e^{M(t-s)}G(y(s))\,ds \\
&\Rightarrow \quad y(t) = (U(t) + V(t))y(0) \\
&\qquad\qquad + \int_{s=0}^{t} (U(t-s) + V(t-s))G(y(s))\,ds \\
&\Rightarrow \quad y(t) = (U(t) + V(t))y(0) + \int_{s=0}^{t} U(t-s)G(y(s))\,ds \\
&\qquad\qquad + \int_{s=0}^{\infty} V(t-s))G(y(s))\,ds - \int_{s=t}^{\infty} V(t-s))G(y(s))\,ds
\end{aligned}$$

To see that the integrals here are all bounded, first note that since y is bounded as $t \to \infty$ (by assumption) and G is continuous, we have that $G(y(s))$ is bounded as $s \to \infty$. Next note that since the real part of the eigenvalues of B are positive, $V(t-s)$ is bounded as $s \to \infty$ (recall that $V(t-s) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & e^{B(t-s)} \end{smallmatrix}\right)$). Continuing,

$$\begin{aligned}
y(t) &= (U(t)+V(t))y(0) + \int_{s=0}^{t} U(t-s)G(y(s))\,ds \\
&\qquad + \int_{s=0}^{\infty} V(t-s))G(y(s))\,ds - \int_{s=t}^{\infty} V(t-s))G(y(s))\,ds \\
\Rightarrow y(t) &= U(t)y(0) + V(t)\left(y(0) + \int_{s=0}^{\infty} V(-s)G(y(s))\,ds\right) \qquad (\star) \\
&\qquad + \int_{s=0}^{t} U(t-s)G(y(s))\,ds - \int_{s=t}^{\infty} V(t-s))G(y(s))\,ds
\end{aligned}$$

Consider the above equation. On the left, we have $y(t)$, which is bounded as $t \to \infty$. One the right, considering the eigenvalues of A and B we see that first, third, and fourth summands are bounded as $t \to \infty$. This implies that

$$V(t)\left(y(0) + \int_{s=0}^{\infty} V(-s)G(y(s))\,ds\right)$$

is bounded as $t \to \infty$. But recall that

$$V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{Bt} \end{pmatrix}$$

where B has $n-k$ eigenvalues, each with positive real parts. Since

$$y(0) + \int_{s=0}^{\infty} V(-s)G(y(s))\,ds$$

is bounded (in fact, constant), this means that

$$V(t)\left(y(0) + \int_{s=0}^{\infty} V(-s)G(y(s))\,ds\right) = 0$$

(Note that the above equation is a product of two matrices. So we cannot conclude that either of the factors is the zero matrix.) From equation $(\star)$, above, it follows that

$$y(t) = U(t)y(0) + \int_{s=0}^{t} U(t-s)G(y(s))\,ds - \int_{s=t}^{\infty} V(t-s))G(y(s))\,ds,$$

as we wanted to show.