Math 322 lecture for Friday, Week 8

GLOBAL STABLE AND UNSTABLE MANIFOLDS

Let E be an open subset of $\mathbb{R}^n$ containing the origin, 0, and let $f\colon E \to \mathbb{R}^n$ be continuously differentiable. Consider the system of differential equations $x' = f(x)$. Suppose 0 is an equilibrium point and that Df_0 has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. By the stable manifold theorem, in a neighborhood of 0 there exists a k-dimensional stable manifold S and an $n-k$-dimensional unstable manifold U. The manifold S is tangent at 0 to the stable space E^s for the linearized system $x' = Df_0(x)$. Similarly, U is tangent at 0 to the unstable space E^u for the linearized system. Further, if $\phi_t(x)$ is the flow for the system, then

$$\lim_{t\to\infty} \phi_t(p) = 0$$

for all $p \in S$ and

$$\lim_{t\to-\infty} \phi_t(p) = 0$$

for all $p \in U$.

Define the *global stable and unstable manifolds* at the equilibrium point 0 by

$$W^s(0) := \cup_{t\leq 0}\phi_t(S)$$

and

$$W^u(0) := \cup_{t\geq 0}\phi_t(U),$$

respectively. Here, for any subset $X \subset E$,

$$\phi_t(X) := \{\phi_t(x)\colon x \in X\}.$$

It turns out that these manifolds (i) do not depend on our choice of local stable and unstable manifolds S and U, (ii) are invariant under ϕ_t, and (iii) for all $p \in W^s(0)$,

$$\lim_{t\to\infty} \phi_t(p) = 0$$

and for all $p \in W^u(0)$,

$$\lim_{t\to-\infty} \phi_t(p) = 0.$$

Remark. There is also version of the stable manifold theorem that applies to equilibrium points where the linearization has eigenvalues with real part equal to zero. It states that if the linearization has k eigenvalues with positive real part, j eigenvalues with negative real part, and $m = n - k - j$ eigenvalues with zero real part, then there are manifolds W^s, W^u, and W^c tangent to stable, unstable, and central spaces, respectively, of the linearization having dimensions k, j, and m, respectively. These spaces are invariant under the flow of the system. See our text, Section 2.7.

Hartman-Grobman theorem

We again consider a system $x' = f(x)$ as above with equilibrium point $x_0 = 0$. (For an arbitrary equilibrium point x_0, just replace x by $x - x_0$.) We again assume the linearized system has no eigenvalues with real part equal to 0. These equilibrium points are called *hyperbolic equilibrium points*. Roughly, the Hartman-Grobman theorem says that in a neighborhood of x_0, the system $x' = f(x)$ and the linearized system $x' = Df_{x_0}(x)$ are qualitatively the same, in a way to be made precise below.

Theorem. (Hartman-Grobman) Let E be an open subset of $\mathbb{R}^n$ containing the origin, and let $f\colon E \to \mathbb{R}^n$ be continuously differentiable with Jacobian matrix Jf. Suppose that 0 is a hyperbolic equilibrium point of the system $x' = f(x)$. Then there exist open neighborhoods U and V of the origin and a homeomorphism (i.e., a continuous bijection with continuous inverse)

$$H\colon U \to V$$

with $H(0) = 0$ having the following property: for all $x_0 \in U$, there is an interval $I \subseteq \mathbb{R}$ containing the origin such that for all $t \in I$,

$$H(\phi_t(x_0)) = e^{Jf(0)t} H(x_0).$$

The theorem says H maps trajectories of the system $x' = f(x)$ to trajectories of the linearized system $x' = Jf(0)x$ in a neighborhood of the origin. (Nonzero equilibria are handled by translating to the origin, as usual.) The proof of the theorem is outlined in Section 2.8 of our text and goes, again, by the method of successive approximations.

Example. Consider the system

$$\begin{aligned} x' &= -x \\ y' &= y + x^2. \end{aligned}$$

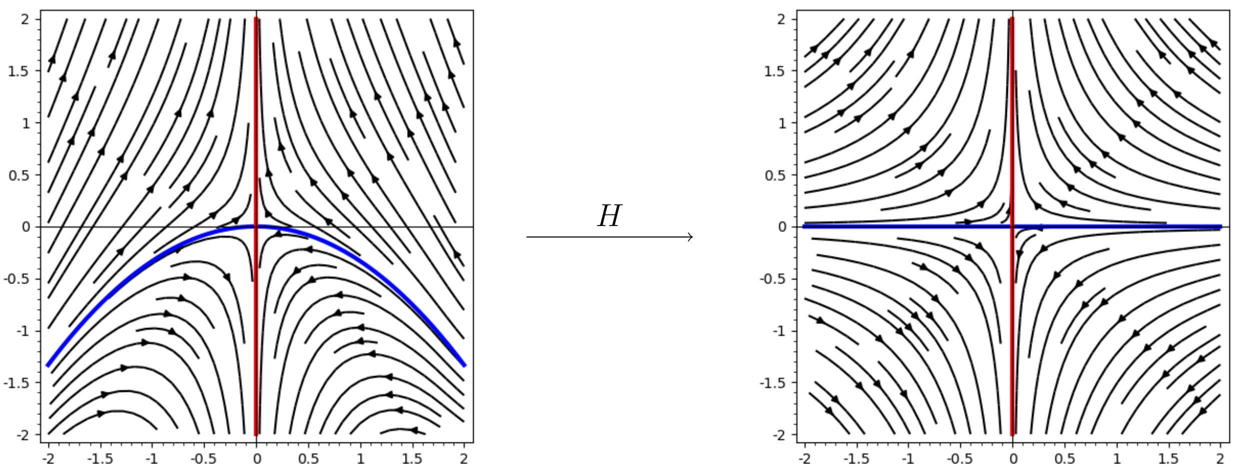
The origin is a hyperbolic equilibrium point, and the linearized system there is

$$\begin{aligned} x' &= -x \\ y' &= y. \end{aligned}$$

Our text shows how to apply the method of successive approximations to find the homeomorphism

$$H(x,y) = \left(x, y + \frac{1}{3}x^2\right).$$

The effect of the mapping is illustrated below with the stable manifolds in blue and the unstable manifolds in red:



The nonlinear system can be solved using the methods we covered during week five of the semester, and the solution with initial condition (x_0, y_0) is

$$x(t) = x_0 e^{-t}$$

$$y(t) = \left(y_0 + \frac{1}{3}x_0^2\right)e^t - \frac{1}{3}x_0^2 e^{-2t}.$$

To find the stable manifold, we find the points (x_0, y_0) such that the solution with that initial points converges to $(0,0)$ as $t \to \infty$. For the unstable manifold, we do the same but with $t \to -\infty$. We find

$$W^s(0,0) = \left\{\left(x, -\frac{1}{3}x^2\right) : x \in \mathbb{R}\right\}$$

$$W^u(0,0) = \{(0,y) : y \in \mathbb{R}\}.$$

The solution to the linearized system is

$$x(t) = x_0 e^{-t}$$
$$y(t) = y_0 e^t$$

with stable and unstable spaces

$$
\begin{aligned}
E^s &= \{(x,0) : x \in \mathbb{R}\} \\
E^u &= \{(0,y) : y \in \mathbb{R}\}.
\end{aligned}
$$

Applying H to the solution of the nonlinear system gives

$$
\begin{aligned}
H(\phi_t(x,y)) &= H\left(xe^{-t}, \left(y + \frac{1}{3}x^2\right)e^t - \frac{1}{3}x^2e^{-2t}\right) \\
&= \left(xe^{-t}, \left(y + \frac{1}{3}x^2\right)e^t\right) \\
&= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} xe^{-t} \\ \left(y + \frac{1}{3}x^2\right)e^t \end{pmatrix} \\
&= e^{Jf(0,0)t}H(x,y).
\end{aligned}
$$

The stable and unstable manifolds for the nonlinear system are mapped by H to the stable and unstable spaces, respectively, for the linear system:

$$
H\left(x, -\frac{1}{3}x^2\right) = (x,0),
$$

and

$$
H(0,y) = (0,y).
$$